Journal of the
OPTICAL SOCIETY *of* AMERICA

VOLUME 31 JULY, 1941 NUMBER 7

Optics of Atmospheric Haze

E. O. HULBURT
Naval Research Laboratory, Washington, D. C.
(Received April 21, 1941)

By means of a telescopic photometer measurements were made in daylight of b, the brightness of a large approximately blackbody at a distance $d=3.2$ km, and of h, the brightness of the sky just above the blackbody. For visual ranges v from 3.2 to 15 km the measurements agreed with the theoretical relations $v=(1/\beta)\log 1/\eta$ and $\beta=(1/d)\log(1/(1-b/h))$, where β is the atmospheric attenuation coefficient, and η is the threshold of brightness contrast, being 0.02 for the usual intensities of daylight illumination. It followed that the ratio of β to the attenuation of optically pure air was 277, 55.4, 27.7, 13.8, 5.54 and 2.77 for $v=1$, 5, 10, 20, 50 and 100 km, respectively. The angular distribution of light scattered by haze was determined in two ways (1) by measuring the brightness at various angles of a searchlight beam at night, and (2) by measuring the brightness of the horizon sky at various azimuths for a moderately low sun and a cloudless sky. The distribution was much the same for haze ranging from thin to thick. It showed pronounced forward scattering, over three-fourths of the light being scattered in a forward direction and less than one-fourth in a backward direction. Such a scattering distribution would occur from haze particles of various sizes, most of them being of dimensions greater than the wave-lengths of white light.

INTRODUCTION

HAZINESS is the normal condition of the atmosphere at sea level and it is probably true that an atmosphere free from optical impurities rarely or never occurs. Various investigations have been carried out dealing with the dependence upon wave-length of the attenuation of light in haze and fog, but there are few quantitative data on the magnitude and other characteristics of the attenuation. It is with this aspect of the subject that the present work has been concerned. In the present experiments the attenuation of light passing through the atmosphere near sea level and the angular distribution of the scattered light have been measured for atmospheric conditions ranging from very clear to foggy. The attenuation was determined from measurements of the brightness of a large approximately blackbody at a distance and of the horizon sky just above the blackbody. The angular distribution of the scattered light was determined from measurements of (1) the brightness at various angles of a searchlight beam at night and (2) the brightness of the horizon sky at various azimuths for a moderately low sun and a cloudless sky.

An excellent survey of the literature of the optics of haze was compiled by Middleton,[1] which discusses most of the references cited below. One is impressed by the scarcity of experimental data.

ATMOSPHERIC ATTENUATION AND VISUAL RANGE

Theoretical

Consider a blackbody at a distance d from the observer viewed against the sky as a background.

[1] W. E. K. Middleton, *Visibility in Meteorology*, (University of Toronto Press, 1935).

BOSTON UNIVERSITY
COLLEGE OF LIBERAL ARTS
LIBRARY

And to be specific without loss of generality, suppose that the blackbody and the observer are on the surface of the earth and that the sky background is the sky near the horizon. In daylight the brightness b of the blackbody as seen by the observer is the flux of light scattered toward the observer by the column of atmosphere along d. Denote distances along the column from the observer by x. Let i erg sec.$^{-1}$ be the flux of daylight incident on a unit volume of the column; in general i is from the sun, sky and surface of the earth. Let σ and β be the scattering and attentuation coefficients per unit volume for light passing along the column to the observer. In general σ is a function of ϕ the angle between the directions of the incident and scattered light; in the present case σ is the fraction of i, no matter how i is distributed in ϕ, in wave-length and in polarization, which is scattered in the direction of $-x$. β refers to rectilinearly propagated light, i.e., non-diffuse propagation, according to

$$i = i_0 e^{-\beta x}, \tag{1}$$

where i_0 is the original intensity and i the intensity after passage through a distance x of the medium. β is due to true absorption and to attenuation by scattering, the first being negligibly small compared to the second for daylight in the normal atmosphere.[2]

It is assumed that i and the optical properties σ and β of the atmosphere are uniform throughout the length of the column from the observer to the blackbody and to infinity. Then the flux toward the observer from an element dx of the column is $\sigma i dx$ erg cm^{-2} sec.$^{-1}$ sterad.$^{-1}$ and the amount which reaches him is $\sigma i e^{-\beta x} dx$. The brightness b of the blackbody is

$$b = \int_0^d \sigma i e^{-\beta x} dx = (\sigma i/\beta)(1 - e^{-\beta d}). \tag{2}$$

The brightness h of the sky background is

$$h = \int_0^\infty \sigma i e^{-\beta x} dx = \sigma i/\beta. \tag{3}$$

From (2) and (3)

$$\beta = (1/d) \log 1/(1 - b/h). \tag{4}$$

[2] E. O. Hulburt, J. Opt. Soc. Am. **25**, 125–130 (1935).

From (4) a measurement of b/h gives the atmospheric attenuation coefficient β. Since β is independent of azimuth, the right-hand side of (4) should be constant with azimuth. This was found to be true experimentally by Löhle[3] using pine-clad hills as targets and by Koschmieder and Rühle[4] using large screens of black velvet. They concluded that various discrepancies which occurred between observation and (4) could reasonably be attributed to lack of uniformity in azimuth of the illumination or of the atmosphere.

The "visual range" v is defined as the maximum distance at which a black object can be seen against the horizon sky background in daylight. The black object should be large enough to subtend an angle greater than about 0.5° at the observer in order to avoid the effects of diffraction. To be precise the black object is on the borderline of invisibility when

$$1 - b/h = \eta, \tag{5}$$

where η is a small quantity. η is the threshold of brightness contrast, and for usual intensities of daylight illumination is about 0.02. This means that the black object is at the visual range when, because of the illuminated air column between it and the observer, its apparent brightness is about 2 percent less than the brightness of the horizon sky background.

At the visual range $d = v$ in (4). Putting this and (5) into (4) yields

$$v = (1/\beta) \log 1/\eta. \tag{6}$$

From (4) and (6)

$$v = d \frac{\log 1/\eta}{\log 1/(1 - b/h)}, \tag{7}$$

which expresses v in terms of directly measurable quantities. With suitable assumptions expressions for the visual ranges of gray, colored and white objects may be readily derived, but are not needed here.

From (1) and (6) it follows that $i = i_0 \eta$, which shows that light is attenuated to η of its original intensity in a distance equal to the visual range. For example, if the atmospheric haze were such

[3] F. Löhle, Met. Zeits. **46**, 49–59 (1929).

[4] H. Koschmieder, *Danziger Sichmessungen* (Forschungsarb. Staatl. Obs. Danzig), No. 2 (1930); H. Rühle, *ibid.*, No. 3.

that the daytime visual range of a black object against the sky were 10 km with $\eta=0.02$, a light would be reduced to 0.02 of its original value by attenuation in traversing 10 km of the atmosphere. Now, lights are often so bright as to be easily visible when reduced in intensity much more than 0.02. Such lights at night could be seen farther than black objects during the day with no change in atmospheric conditions. This is a very familiar circumstance, particularly to navigators, who often may not be able to see a shore during the day because of obscuration by the atmosphere and yet after nightfall are able easily to see lights on the shore through the same obscuration.

It is of interest to compare the atmospheric attenuation β with the attenuation β_0 of pure air. Since the degradation of light of the visible part of the daylight spectrum in pure air is due almost entirely to scattering by the air molecules, β_0 was calculated from the Rayleigh formula

$$\beta_\lambda=32\pi^3(\mu_\lambda-1)^2/3n\lambda^4, \qquad (8)$$

where β_λ is the attenuation of light of wavelength λ, μ_λ is the refractive index and n the molecular density of air. If i_λ is the energy of daylight in the interval from λ to $\lambda+d\lambda$

$$\beta_0=\int\beta_\lambda i_\lambda d\lambda/\int i_\lambda d\lambda, \qquad (9)$$

the integrals being taken over the visible spectrum from 4000 to 7400A. The spectral quality i_λ of daylight is a close approximation to that of solar radiation outside of the atmosphere regardless of the state of the weather or the altitude of the sun.[5] Then with i_λ and μ_λ from standard tables[5,6] (9) was evaluated and gave $\beta_0=0.0141$ per km of air at a temperature of 0°C and a pressure of one atmosphere.

In Table I are given values of v and related quantities calculated from the attenuation coefficient β for various atmospheric conditions. v was calculated from (6) with $\eta=0.02$, b/h from (4) with $d=3.2$ km, and the ratios β/β_0 with $\beta_0=0.0141$ per km. Since, as described in the next section, a portion of the table from $v=0.6$ to 15 km is in agreement with observation, the table may be regarded as a fairly correct statement of the relations between various optical constants of the atmosphere under a variety of conditions.

TABLE I. *Visual range and related quantities.*

β PER KM	b/h FOR $d=3.2$ KM	v, $\eta=0.02$ KM	v, $\eta=0.02$ NAUTICAL MILE	β/β_0	ATMOSPHERIC CONDITION
42.6	—	0.09	0.05	3030	Dense fog
21.3	—	0.18	0.10	1510	
7.13	—	0.55	0.30	505	
4.27	—	0.92	0.50	303	Fog
2.14	—	1.83	1.00	151	
1.42	—	2.7	1.50	101	Light fog
1.22	0.98	3.2	1.75	86.5	
1.07	0.96	3.7	2	75.6	
0.533	0.82	7.3	4	37.8	Haze
0.354	0.67	11.0	6	25.1	
0.268	0.57	14.6	8	19.0	
0.214	0.50	18.3	10	15.1	Clear
0.142	0.37	27.4	15	10.1	
0.107	0.29	36.6	20	7.56	
0.0713	0.22	54.9	30	5.05	Very clear
0.0427	0.14	92	50	3.03	Exceptionally clear
0.0304	0.09	128	70	2.16	
0.0214	0.07	183	100	1.51	
0.0171	0.05	229	125	1.21	
0.0141	0.04	277	151	1.00	Pure air

Experimental

The opening in the end of a large hangar situated at a distance of 3.2 km from the observer served as an approximately blackbody. Thus d of (7) was 3.2 km. The opening was 30 feet high and about 100 feet long, and subtended an angle of about 0.17° in height and 0.5° in width at the observer. The end of the building, painted a medium gray color, extended from 0.1° to 0.2° on each side and above the opening. About two km behind the hangar were low hills rising about 0.2° above the hangar, the opening being therefore about 0.3° below the horizon sky. It has been calculated[7] that a black target against the sky should subtend a cone of angular diameter greater than about 0.5° at the observer in order to avoid the seepage of light into the cone by diffraction. In the present case although the target was somewhat smaller than this in height no diffraction effects were observable because the target was surrounded by areas considerably darker than the sky. The line of sight from the observer to the hangar ran in a northwesterly direction along the valley of the Potomac River about 4 km from the city of Washington.

A telescopic photometer was mounted in a dark room and was pointed at the hangar scene

[5] A. C. Hardy, *Handbook of Colorimetry* (Massachusetts Institute of Technology Press, 1936), p. 18.
[6] *Smithsonian Physical Tables* (1933), Table 413.
[7] Reference 1, page 34.

through a 6-inch hole in the side of the room. The photometer, shown in Fig. 1, consisted of an objective lens L, an eyepiece E and a photometer part P. P was a Macbeth illuminometer, Leeds and Northrup Company, mounted in the telescope EL with the photometric cube in the focal plane of L. A blue filter F was placed over the comparison lamp of the illuminometer to obtain a fair color match with the hangar and the horizon sky. Lens L was 8 cm in diameter and

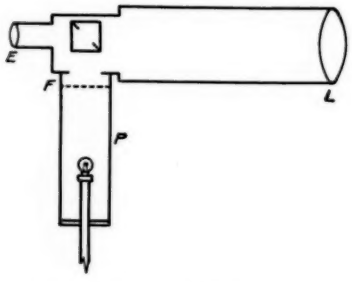

FIG. 1. Telescopic photometer.

62 cm focal length; its surfaces were coated with low reflecting films of magnesium fluoride. The telescopic photometer was satisfactory to operate due in part to the reduction of scattered light in the instrument occasioned by the low reflecting films and in part to the fact that the instrument and the observer were in a dark room which made for excellent working conditions. Brightness measurements could be repeated within 3 to 5 percent and the b/h ratios within about 7 percent.

The correction of b due to the fact that the hangar opening was not a blackbody may be expressed by adding a term $b_0\epsilon^{-\beta d}$ to the right-hand side of (2), where b_0 is the brightness of the hangar opening at zero distance; for a blackbody $b_0=0$. The correction was neglected, for direct measurement of the brightness b_0 of the rear wall inside the hangar opposite the opening, showed that the term contributed less than 3 percent to b in weather for which v was less than 15 km.

The visible range v was determined from various objects, as groves of trees, hills and buildings of the city, the most distant object being the wooded skyline 15 km away. Values of b/h and v were recorded for various atmospheric conditions of clear, haze, fog, drizzle, rain and snow, but only when the sky was either completely cloudless or completely overcast in order to insure as far as possible a uniform illumination along the line of sight. Scattered clouds gave irregular illumination and erratic values of b/h. About a tenth of the observations, selected to be fairly representative of the total number, are plotted as circles and dots in Fig. 2, the circles and dots referring, respectively, to a clear and a cloudy sky. The continuous curve of Fig. 2 was calculated from (7) with $\eta=0.02$, and the dotted curves from (7) with $\eta=0.01$ and 0.05. The curve for $\eta=0.02$ is the plot of columns 2 and 3 of Table I; β/β_0 of Table I is marked on the right side of Fig. 2.

It is seen that the observations are in fair accord with the theory with the threshold brightness contrast ratio $\eta=0.02$. b/h values during rain and snow, and during haze whether due to smoke, dust or moisture, also fell in with the curve of $\eta=0.02$. In fact it was soon evident that the quantity most uncertain of exact measurement was v. It turned out that the observer had a tendency to underestimate v, due partly to the fact that the objects used to estimate v were not black and could have been seen farther if they had been black, and partly to the fact that the observer was inclined to "play safe" and record the visual range by what he could see and not by what he could barely not see. An ideal system of targets would have helped matters, as a series of very large black objects placed at each kilometer out to 30 km or more; such a system was unobtainable

During the period of observation the lowest value of b/h encountered was 0.22. From (5) or Table I this would correspond to a visual range of 52 km. The occasion was during a cold clearing after a winter rain before the northwest wind had set in in earnest. The air was very clear, but even so its attenuation was five times that of pure air.

In the present case no b/h measurements could, of course, be made for v less than 3.2 km, for when v was less than this the hangar could not be seen. As a test of the theory for a range below 3.2 km it may be recalled that the visible portion of the spectrum of a light was observed[8]

[8] L. P. Granath and E. O. Hulburt, Phys. Rev. **34**, 140–144 (1929).

to be decreased to 0.12 of its clear air intensity after passage through 400 meters of fog so thick that v was 0.6 km. Putting $i/i_0=0.12$ into (1) gave $\beta=5.34$, and this in (6) yielded $v=0.73$ km, which is in fair agreement with the observed value 0.6 km.

It is concluded that the theory embodied in Eqs. (1) to (7) is in agreement with experiment and therefore that Table I expresses facts regarding certain optical quantities of the atmosphere. The remark may now be made that the theory is so simple that it has needed no experimental confirmation anyway, and that the experimental confirmation that has been secured amounts merely to showing that light is attenuated in the atmosphere according to an exponential law and that a two percent difference in the brightness of two sizable objects in daylight is about the limit of distinguishability, these things being well enough known already. The remark is certainly true, and about the only answer to it is that the experiments have placed the theory on a firm foundation.

In the light of the present results on atmospheric attenuation it is of interest to discuss the Luminous Range Table of the Bureau of Lighthouses.[9] The distance at which the light of a lighthouse can be seen depends mainly upon the curvature of the earth, the brilliancy of the light and the attenuation of the atmosphere. The Luminous Range Table takes into consideration the last two factors and gives the brilliancy i_0 in candlepower of the light for the designed range x_r nautical miles. The lights of United States lighthouses are built to the specifications of the Table, or to exceed the specifications. It may be shown that the values of the Table are expressed by

$$i_0=0.283x_r^2 10^{\beta x_r}, \qquad (10)$$

where the atmospheric attenuation coefficient β is 0.0882 for x_r in nautical miles and is 0.111 for natural logarithms and x_r in km. This means an atmosphere for which β/β_0 is 8.0 and the visual range v is 35 km or 19 nautical miles. At the distance x_r the illumination from the light is 0.79×10^{-8} footcandle. The illumination from a star of the fourth magnitude is 0.53×10^{-8} footcandle. Thus at the designed limiting range the brightness of the lighthouse light appears about as bright as a fourth magnitude star (which is not very bright) when the atmospheric conditions are such that v is 19 nautical miles (which is a fairly clear atmosphere). When v is greater and less than 19 miles the actual range of the light will, of course, be greater and less, respectively, than the designed range.

Also, it is of interest to provide a basis for the International Visibility Code in terms of the atmospheric attenuation of daylight. This is done in Table II in which the first four columns are from the visibility code used by practically all meteorological services.* In the fifth and sixth columns of Table II are given the values of β and β/β_0 corresponding to v in English units of the third column calculated from (6) with $\eta=0.02$ and $\beta_0=0.0141$.

Angular Distribution of the Light Scattered by Haze

Theoretical distribution curves

(a) *Particles of dimensions small with respect to the wave-length.*—In this case for unpolarized light

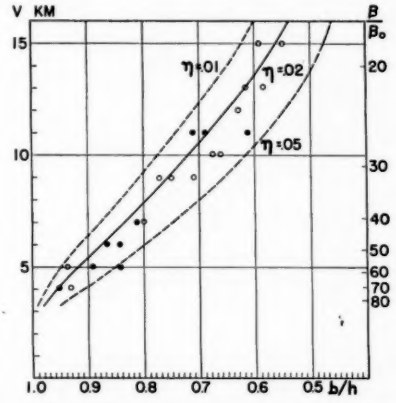

FIG. 2. Circles and dots observed values of b/h in sunny and cloudy weather, respectively; smooth curves theoretical.

[9] Reference 2, page 126, gives a portion of the Luminous Range Table.

* I am indebted to Dr. I. R. Tannehill of the United States Weather Bureau and to Dr. Robert G. Stone of the Department of Meteorology of New York University for the International Visibility Code. See also reference 1, page 74. The words in column 4 of Table II are not agreed to by every one. This may be expected; for example, a fog of visual range 400 yards might be termed "moderate" by an observer on land, and "dense" or "thick" by an observer at sea.

incident on the particle the flux σ_ϕ per unit solid angle scattered in the direction at angle ϕ to the direction of propagation of the incident light is

$$\sigma_\phi = \sigma_1(1+\cos^2\phi), \tag{11}$$

where σ_1 is the value of σ_ϕ for $\phi=\pi/2$. The σ_ϕ/σ_1 curve of (11) is plotted in curve 1, Fig. 3 with the ordinate at $\phi=0$ arbitrarily made unity. It is seen that the forward and backward scattering are symmetrical and equal.

(b) *Particles of dimensions comparable with the wave-length.*—Rayleigh[10] worked out this case for unpolarized light incident on dielectric smooth spheres of refractive index 1.5 and of radii $r=\lambda/6.28$, $\lambda/4.16$, $\lambda/3.59$ and $\lambda/3.14$. The amplitudes ζ and ξ of the two polarized components at right angles to each other were tabulated for various values of ϕ. (In Rayleigh's notation $\zeta=$modulus $(yZ-zY)/r$ and $\xi=$modulus Y.) The scattered intensity is proportional to $\zeta^2+\xi^2$. The σ_ϕ/σ_1 curves calculated from this expression are plotted in curves 2 to 5, Fig. 3, with unit ordinate at $\phi=0$.

It would be of interest to have similar curves for a refractive index 1.33, corresponding to water, but the task of computation has not been undertaken.

TABLE II. *International visibility code and atmospheric attenuation.*

Code No.	Daylight Visual Range v		Weather Fog, Mist, Haze	β per km, Nat. Log.	β/β_0
	Metric	English			
0	<50 m	<50 yards	dense fog	>85.6	>6060
1	50 200	50 200	thick fog	85.6 21.4	6060 1515
2	200 500	200 500	moderate fog	21.4 8.54	1515 606
3	500 1000	500 1000	light fog	8.54 4.27	606 303
4	1 km 2	½ naut. 1 mile	thin fog	4.27 2.14	303 151
5	2 4	1 2	haze	2.14 1.07	151 75.6
6	4 10	2 5	light haze	1.07 0.427	75.6 30.2
7	10 20	5 10	clear	0.427 0.214	30.2 15.1
8	20 50	10 30	very clear	0.214 0.0713	15.1 5.05
9	>50	>30	exceptionally clear	<0.0713	<5.05

[10] Rayleigh, Proc. Roy. Soc. **A84**, 25–46 (1911).

(c) *Particles of dimensions large with respect to the wave-length.*

(1) Large smooth dielectric sphere.—An unpolarized ray incident on the sphere is in general divided into two rays, one which is reflected from the surface of the sphere and one which passes through the sphere with refraction at the surfaces where it enters and emerges. Considering the reflected ray the fraction σ_ϕ' of the energy intercepted by the sphere diverted in the direction ϕ is per unit solid angle

$$\sigma_\phi' = \rho_i/4\pi, \tag{12}$$

where $\phi=2\pi-i$ and the reflectivity ρ_i at incident angle i is given by Fresnel's equations. In the case of the refracted ray the fraction σ_ϕ'' of the energy intercepted by the sphere diverted in the direction ϕ is per unit solid angle

$$\sigma_\phi'' = \frac{\sin 2i}{(4\pi\sin\phi)(1-\cos i/\mu\cos r)}, \tag{13}$$

where $\phi=2(i-r)$ and $\sin i=\mu\sin r$, μ being the refractive index. Adding σ_ϕ' and σ_ϕ'' gives the total fraction σ_ϕ diverted in the direction ϕ. The σ_ϕ/σ_1 curves for refractive indices 1.5 and 1.33 are in Fig. 3 curves 6 and 7; they show pronounced forward scattering. Curve 7 for $\mu=1.33$ refers to spheres of water, as rain drops. The sum of the energies of (12) and (13) integrated over the sphere is 0.97 of the energy intercepted by the sphere; the remaining 0.03 forms n rainbows.

It is seen from curves 1 to 6, Fig. 3, that the forward scattering increases as the ratio r/λ increases from 0 to about $\frac{1}{3}$, and then continues to increase only very slightly as r/λ increases from $\frac{1}{3}$ to infinity. This conclusion, which is derived for spheres of refractive index 1.5, is approximately true for spheres of refractive index 1.33, since curves 6 and 7, Fig. 3, are not greatly different.

(2) Large rough sphere.—In the case of a large and perfectly rough sphere[11] the surface of which obeys Lambert's cosine law

$$\sigma_\phi = (4/3\pi)(\sin\phi-\phi\cos\phi). \tag{14}$$

The σ_ϕ/σ_1 curve from (14) is curve 8 in Fig. 3; it

[11] Equation (14) was worked out for me by Mr. J. Sweer who got it in class lectures of Professor A. G. Worthing at the University of Pittsburgh.

shows strong backward scattering. This is the case of snow and large particles of dust. (14) is also the variation of the intensity of moonlight with the phase of the moon.

We conclude this section with the remark that irregularly shaped particles, if uniformly distributed in space orientation, will on the average scatter light as spheres. Therefore the conclusions which have been reached for spheres are valid for particles which are not spheres provided all possible orientations are equally probable, and hence are valid for most of the particles of atmospheric haze.

Intensity of searchlight beam

The angular distribution of the light scattered by the atmosphere was determined by measuring at night the intensity of the beam of a searchlight at various angles. Intensity measurements were made with the Macbeth illuminometer with the blue filter *F*, Fig. 1, on a foggy night and on a clear night. The observed intensities relative to the intensity σ_1 at $\phi=\pi/2$, multiplied by $\sin\phi$ to reduce to equal thickness of beam, are plotted in Fig. 4 as circles and crosses, respectively, the ordinates being σ_ϕ/σ_1. The weather on the foggy night would be described as a light fog or a damp haze. The visual range in the afternoon was about 5 km, and was assumed to remain constant after nightfall since there was no breeze or change in cloudiness; after nightfall lights at 6 km could not be seen. The weather on the clear night was calm and fairly clear; a b/h measurement in the afternoon gave a visual range of about 20 km.

The circles and crosses of Fig. 4 have the same variation with ϕ within the error of experiment, showing that the angular distribution of scattered light was approximately the same for atmospheric conditions varying from fairly clear to moderately foggy. Therefore the smooth curve 1, Fig. 4, may be taken to be representative of many atmospheric conditions. The curve is also drawn in curve 9, Fig. 3, for comparison with the theoretical curves. It is seen that the atmospheric haze curve would agree with the composite curve of a haze composed of a fairly large proportion of water droplets of r/λ between $\frac{1}{3}$ and infinity to account for the forward scattering, together with a fairly small proportion of rough particles of r large with respect to λ to account for the moderate amount of backward scattering. Such a composition of haze seems reasonable, and is probably what haze usually consists of.

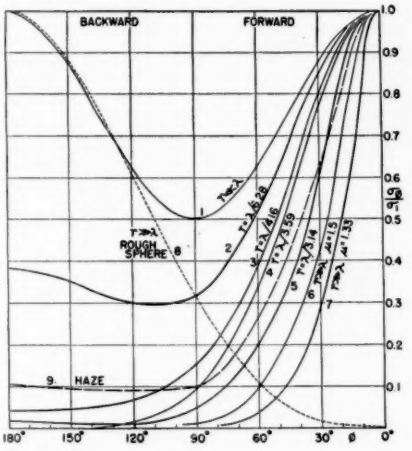

FIG. 3. Curves 1 to 8 theoretical scattering by several types of particles; curve 9 scattering of haze from searchlight observations.

The scattering by the air in an ordinary quiet laboratory room, mainly due to dust, is shown by the dotted curve 2, Fig. 4, the maximum ordinate being made equal to that of the haze curve 1 for comparison. The dust curve lies along curve 1 for $\phi<60°$. It shows considerable backward scattering which would be accounted for by the many large notes which could be seen floating about in the beam of light. A measurement of the intensity of the beam showed that the attenuation of the air in the room was about 20 times that of pure air, which would correspond to a visual range of about 11 km. In other words, if the outside atmosphere were as dusty as the air in the room the visual range would have been 11 km. However, the dust content of the air of the room fluctuated over wide limits with changing conditions.

Brightness of horizon

If the sun is moderately low and if there are no clouds in the sky the angular distribution of the light scattered by the atmosphere near sea level may be derived from observations of the brightness of the sky near the horizon at all azimuths around the compass. Let x denote the distance along a horizontal column of the atmos-

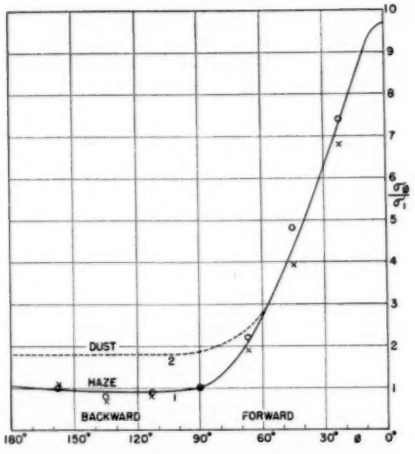

FIG. 4. Scattering observed with searchlight; curve 1 average through circles and crosses observed in foggy and clear atmosphere, respectively; curve 2 dust in air of room.

phere from the observer to the sky near the horizon in a direction ϕ to the sun and let h_ϕ be the sky brightness in this direction. Let i_s erg cm^{-2} sec.$^{-1}$ be the intensity of sunlight and $b_k d\omega$ and $b_g d\omega$ erg sterad.$^{-1}$ sec.$^{-1}$ be the intensity of light from an angular area $d\omega$ of the sky and the ground, respectively. The scattering and attenuation coefficients σ_ϕ and β of the atmosphere are due to the combined effects of the haze particles and air molecules. The amount of sunlight scattered toward the observer by a portion dx of the column of atmosphere is $\sigma_\phi i_s dx$, and of sky light and ground light are $dx \int_U \sigma_\phi b_k d\omega$ and $dx \int_L \sigma_\phi b_g d\omega$, where U and L are the integrals over the upper and lower hemispheres, respectively. The light which reaches the observer is the sum of these multiplied by $e^{-\beta x}$ and h_ϕ is the integral of this from 0 to ∞. The curvature of the earth is neglected. Then

$$h_\phi = \int_0^\infty \left[\sigma_\phi i_s + \int_U \sigma_\phi b_k d\omega + \int_L \sigma_\phi b_g d\omega \right] e^{-\beta x} dx. \quad (15)$$

Uniform illumination and uniform optical properties of the atmosphere, sky and ground along x are assumed. Then β and the expression in brackets are independent of x, and (15) becomes

$$h_\phi = \frac{1}{\beta} \left[\sigma_\phi i_s + \int_U \sigma_\phi b_k d\omega + \int_L \sigma_\phi b_g d\omega \right]. \quad (16)$$

Assuming that the attenuation β is due to scattering only, true absorption being zero, we have

$$\beta = 2\pi \int_0^\pi \sigma_\phi \sin \phi d\phi, \quad (17)$$

which is written

$$\beta = \alpha \sigma_1, \quad (18)$$

where

$$\alpha = 2\pi \int_0^\pi (\sigma_\phi / \sigma_1) \sin \phi d\phi, \quad (19)$$

σ_1 being the value of σ_ϕ at $\phi = \pi/2$.

It is assumed that average values of b_k and b_g may be employed, as though b_k and b_g were constant at all points of the sky and ground, respectively. Then, noting that $d\omega = 2\pi \sin \phi d\phi$,

$$\int_U \sigma_\phi b_k d\omega = \pi b_k \int_0^\pi \sigma_\phi \sin \phi d\phi = \beta b_k / 2, \quad (20)$$

and

$$\int_L \sigma_\phi b_g d\omega = \pi b_g \int_0^\pi \sigma_\phi \sin \phi d\phi = \beta b_g / 2. \quad (21)$$

The use of average values of b_k and b_g is justifiable since the terms containing them are relatively small. True, b_k could have been observed at each point in the sky and (20) integrated with precision, but there was little advantage in doing this since b_g, which was nearly as large as b_k, could not be determined exactly without a complete exploration of the reflectivity of the ground over all areas within 20 km or more of the observer. Introducing (18), (20) and (21) into (16) leads to

$$\frac{\sigma_\phi}{\sigma_1} = \frac{\alpha}{i_s} \left(h_\phi - \frac{b_k + b_g}{2} \right). \quad (22)$$

Since (22) contains the integral expressed by α of (19) a direct solution requires trials with several sets of observed quantities. It is simpler, and amounts to the same thing in the end, to evaluate α from the σ_ϕ/σ_1 curve of the searchlight observations. From curve 1, Fig. 4, α is found to be 26.9. (22) then becomes

$$\frac{\sigma_\phi}{\sigma_1} = \frac{26.9}{i_s} \left(h_\phi - \frac{b_k + b_g}{2} \right). \quad (23)$$

Measurements of h_ϕ, i_s, b_k and b_g were made with the Macbeth illuminometer with the blue

filter over the photometer lamp. The field of view was a cone of about 4° angular diameter. The instrument was carefully calibrated with a standard tungsten lamp so that when pointed at a luminous area it read the energy flux from the area in erg sec.$^{-1}$ sterad.$^{-1}$ of light in the daylight spectrum[5] from 4000 to 7400A. The flux in ergs could be reduced to millilamberts by dividing by 35.5. h_ϕ was measured by pointing the illuminometer directly at the sky about 5° above the horizon, but values could not be taken closer to the sun than about 10°, because direct sunshine in the instrument caused errors. i_s was determined by measuring the energy flux from a white plaque of known diffuse reflectivity normal to the sun's rays and protected from the sky light, and b_k from the plaque horizontal and exposed to the sky and screened from the direct sun. A reading was also taken of the horizontal plaque exposed to both the sky and the sun. Multiplying this reading by the reflectivity of the ground gave b_g. The observed reflectivity of winter woods was about 7 percent, of vegetation-covered fields about 10 percent and of ploughed land about 15 percent. Because of gloss the reflectivity of grassy fields was much greater than 10 percent for angles of incidence above 70°. From all this the average reflectivity of the Maryland countryside in winter was taken to be 10 percent for dry weather and no snow. The simple theory underlying the above measurements need not be detailed here, nor the calibration of the instrument.

Many series of values of h_ϕ around the horizon were recorded for altitudes of the sun between 10° and 30°, the sky entirely free of clouds and the visual range greater than about 2 km. If the haze were too thick or the sun too low $(b_k+b_g)/2$ was larger than a few percent of i_s and might have introduced uncertainties. A typical series of measurements is given in Table III. The values of h_ϕ in the second column are averages of values in the two semi-circles on either side of the bearing of the sun, otherwise they have not been smoothed. Since 10 to 15 minutes were required to make each series of h_ϕ observations, the values of i_s, b_k, b_g and the altitude of the sun were averages of measurements taken at the beginning and end of the h_ϕ series.

TABLE III. *Horizon sky data.*

Altitude of sun, 25°:
$v=9$ km
$i_s=52.0\times10^4$ erg cm^{-2} sec.$^{-1}$
$b_k=0.96\times10^4$ erg sec.$^{-1}$ sterad.$^{-1}$
$b_g=0.61\times10^4$ erg sec.$^{-1}$ sterad.$^{-1}$

ϕ	h_ϕ	σ_ϕ/σ_1
20°	16.3×10^4	8.04
29°	10.9	5.23
38°	7.65	3.55
50°	6.18	2.79
65°	3.86	1.59
79°	2.93	1.11
92°	2.62	0.95
101°	2.47	0.87
128°	2.62	0.95
146°	2.93	1.11
150°	3.14	1.22

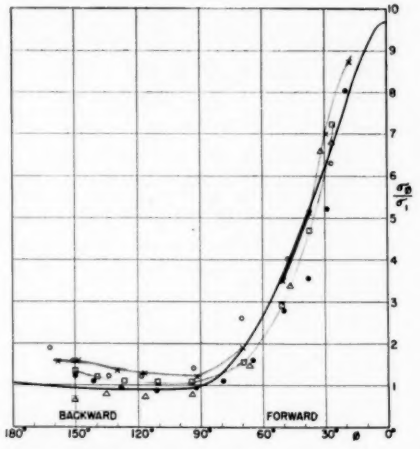

FIG. 5. Scattering by atmosphere; smooth curve from searchlight data; triangles, dots, circles, crosses and squares from horizon sky data for $v=3$, 9, 12, 17 and 40 km, respectively.

In the third column of Table III are the values of σ_ϕ/σ_1 calculated from (23); these are plotted as dots in Fig. 5. In Fig. 5 are also plotted four other representative series of values of σ_ϕ/σ_1 which cover a wide range of atmospheric conditions. The values of v were 3, 9, 12, 17 and 40 km for the triangles, dots, circles, crosses and squares, respectively, the first four values of v being from direct observation and the last from a b/h measurement. The data of the triangles and squares were obtained within 5 km of Washington, the first referring to a relatively dry quiet uniform winter haze, probably mostly smoke, and the second to a fairly clear atmosphere. The dots, circles and crosses of Fig. 5 were from data

obtained in the open country more than 50 km from the nearest large city, the atmospheric condition appearing to be a true moisture haze. It is to be noted that the points of Fig. 5 were calculated entirely from observed data with no arbitrary constants.

The smooth curve in Fig. 5 is curve 1, Fig. 4, of the searchlight observations. It is in fair agreement with the horizon sky data, showing that the results from the two different types of experiments are much the same. Therefore, curve 1, Fig. 4, may be taken to represent tolerably well the angular distribution of the light scattered by the atmosphere for a considerable variety in type and condition of atmospheric haziness. The variances of the series of points of Fig. 5 from the smooth curve and from each other are, however, real and outside the errors of the optical measurements. They are to be ascribed partly to true differences in the optical properties of the various hazes, and partly to non-uniformity of haze conditions with azimuth, non-uniformity of sky and earth illumination and to other relatively minor differences between theoretical assumption and condition of experiment. A noticeable case of such a difference occurred during observations on the shore of Chesapeake Bay in calm weather. Toward the sun the horizon sky was considerably brighter than in similar conditions over land, due to the additional light from the water reflection.

The point may now be made that although the σ_ϕ/σ_1 values from the horizon measurements have seemingly depended on the searchlight data because of the use of curve 1, Fig. 4, in obtaining 26.9 for the integral α of (19), this is really not the case. For one can use the horizon data themselves to derive α, and in so doing come out with approximately the same value 26.9.

The slight increase in h_ϕ and σ_ϕ/σ_1 as ϕ increased from 90° to 180°, visible in Table III and the points of Fig. 5, was real and was noted many times. To elucidate this small effect and others would require a more complete theoretical and experimental investigation than has been attempted here. Further, although curve 1, Fig. 4, is approximately correct for a number of atmospheric conditions it is not true for all conditions, as, for example, snow which may be expected to give much stronger backward scattering than haze. Another example occurred during a very windy dry cold spell, when the lower atmosphere became filled with a dust haze, v being 4 or 5 km. In this case the horizon sky opposite the sun was much brighter than that given by curve 1, Fig. 4. It was similar to curve 2, Fig. 4, indicating that the atmospheric haze was composed of large opaque particles which caused considerable backward scattering.

The Matrix Algebra and Color Reproduction

CARL W. MILLER
Brown University, Providence, Rhode Island
(Received April 17, 1941)

The color of an object is characterized by three reflection densities, measured through three primary color filters. A set of subtractive primaries is, therefore, characterized by a color matrix (c) in which the elements are the densities of unit concentrations of the primaries. The vector density which characterizes a reproduction is obtained from the vector density of the subject by operating on it with the matrix (c). Two successive reproductions are characterized by the matrix product of the two individual matrices. The character of the masking which is required for correct reproduction is determined by the inverse matrix (c^{-1}). A method is described for appraising the results to be achieved by any scheme of partial masking, and these results are applied to successive reproductions such as are involved in reproducing a Kodachrome transparency. The method is applied to calculating the masking which is required in a reproduction to compensate for errors which are anticipated in a subsequent reproduction. The extent to which these matrix calculations are vitiated by the non-additivity of the constituent densities is discussed.

FOR the purposes of photographic reproduction, the color of an object is most conveniently characterized by reflection densities, measured through three primary color filters. Let us imagine the color area in question to be photographed through the three primary filters in conjunction with a neutral gray scale, and let us suppose that three positive reproductions have been made in such a fashion as to yield an over-all gamma of unity, thus providing correct and identical renditions of the gray scale in each of the positives. The densities of the color area in these three reproductions will constitute such a measure of the color. If the over-all characteristics of the filters and the sensitive emulsion are linear transforms of the ICI standard observer curves, the densities so obtained will be satisfactory measures of color, in the sense that all colors which appear alike to the eye will be represented by the same densities, and all colors which possess the same densities will appear alike to the eye.

If a color record of any subject is made through three such ideal filters, the problem of correctly reproducing the original is simply that of synthesizing colors which will have the same three densities. The over-all characteristics of practical taking filters and emulsions, however, are less overlapping than the ICI curves, and hence do not yield a perfect analysis of the subject. Even in such cases, nevertheless, the best which can be done in general is to synthesize colors which will have the same densities as those yielded by the actual filters. This method of approach to the problem of color reproduction has been discussed in detail by Yule[1] and Marriage.[2] It is the purpose of the present paper to outline a mathematical technique of considerable power and elegance for use when this method of approach is adopted.

It is customary to synthesize the colors by superposing three subtractive primaries, cyan, magenta and yellow. Unit concentration of each will be defined as that which has a density to the complementary taking filter of unity. Unfortunately each subtractive primary absorbs not merely its own complementary color but some of the others as well. Each primary is, therefore, characterized by three densities, the density of unit concentration to each of the three taking filters. The set of primaries is, accordingly, characterized by nine densities, which are quite naturally arranged as a square matrix.

$$(c)=\begin{pmatrix} c_{1x} & c_{2x} & c_{3x} \\ c_{1y} & c_{2y} & c_{3y} \\ c_{1z} & c_{2z} & c_{3z} \end{pmatrix}. \qquad (1)$$

Here the subscripts 1, 2, 3, are used to refer to the three subtractive primaries, cyan, magenta and yellow, respectively. The subscripts x, y, z,

[1] J. A. C. Yule, J. Opt. Soc. Am. **28**, 419–430, 481–492 (1938).
[2] A. Marriage, Phot. J. **80**, 364–374 (1940).

refer to the red, green, and blue taking filters. Thus c_{1x}, c_{2y}, and c_{3z}, representing the density of unit concentration of cyan to red light, magenta to green light, and yellow to blue light, are by definition all unity. This matrix will be referred to as the "unit concentration" matrix or simply as the "color" matrix. It plays much the same role in color photography which "gamma" plays in monochrome work.

Since color is characterized by three densities, it is a vector quantity, and this vector density will be represented by d. The components of d, i.e., the densities to the three primary filters, will be represented by x, y, z. Similarly the vector density of the reproduction will be represented by D, and its components by X, Y, Z. Consider now the effect of superposing three subtractive primaries (cyan, magenta and yellow) which have densities (to the complementary color filter) of x_1, y_2 and z_3, respectively. This group of three primary densities we will represent by the vector D_0. We will assume tentatively that the total density to each of the three taking filters is equal to the sum of the three individual densities, but will return to a consideration of this hypothesis at the end of the paper. With this assumption of additivity of densities,

$$\begin{aligned} X &= c_{1x}x_1 + c_{2x}y_2 + c_{3x}z_3, \\ Y &= c_{1y}x_1 + c_{2y}y_2 + c_{3y}z_3, \\ Z &= c_{1z}x_1 + c_{2z}y_2 + c_{3z}z_3. \end{aligned} \tag{2}$$

These component equations will in general be summarized by a single matrix equation.

$$D = (c)D_0. \tag{3}$$

The vector density of the reproduction is, therefore, obtained from the vector density of the primary color constituents through the agency of the color matrix (1). All of the early attempts at the photographic reproduction of color were directed toward the realization of primaries for which all the elements of (c) should be zero except those in the main diagonal. If such primaries could be secured, correct color reproduction would merely involve laying down the individual constituents in such a manner that $x_1 = x$, $y_2 = y$, and $z_3 = z$. It is now apparent from fundamental considerations that such primaries are impossible to achieve, and present-day efforts at precision reproduction seek to utilize in forming each element, not merely the record obtained with the complementary filter, but that of the others as well. This is done by so-called "masking" or by photoelectric scanning methods. Let us suppose that x_1 is obtained by the superposition of three constituents, made from the three filter records with gammas of m_{1x}, m_{1y} and m_{1z}, respectively, and that y_2 and z_3 are similarly determined. Then

$$\begin{aligned} x_1 &= m_{1x}x + m_{1y}y + m_{1z}z, \\ y_2 &= m_{2x}x + m_{2y}y + m_{2z}z, \\ z_3 &= m_{3x}x + m_{3y}y + m_{3z}z. \end{aligned} \tag{4}$$

Summing up as before in a single matrix equation,

$$D_0 = (m)d \tag{5}$$

where (m), to be known as the "masking" matrix, has for its elements the various gammas to which the individual records are reproduced.

$$(m) = \begin{pmatrix} m_{1x} & m_{1y} & m_{1z} \\ m_{2x} & m_{2y} & m_{2z} \\ m_{3x} & m_{3y} & m_{3z} \end{pmatrix}. \tag{6}$$

Substituting D_0 from (5) in (3),

$$D = (c)(m)d. \tag{7}$$

It is a familiar fact of the matrix algebra that two such successive operations as indicated by (7) are equivalent to a single operation which is characterized by the matrix product (cm). The values of the elements of (cm) are obtained by the simple and familiar rule for finding a matrix product.

The transformation (7) will be an identical transformation and correct reproduction will be achieved if $(cm) = I$, where I is the identical matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

The necessary condition for this is that $(m) = (c^{-1})$, i.e., if the masking matrix is the inverse of the color matrix. A typical (c) matrix for dyes commonly used in photography is the following.[3]

$$(c) = \begin{pmatrix} 1 & 0.14 & 0.02 \\ 0.52 & 1 & 0.08 \\ 0.32 & 0.45 & 1 \end{pmatrix}. \tag{9}$$

[3] I am indebted to Dr. Eduard Roosen-Runge of the Brown University Photographic Laboratory for the measurements on which these matrices are based.

For this set of primaries,

$$(c^{-1}) = \begin{pmatrix} 1.08 & -0.15 & -0.01 \\ -0.54 & 1.11 & -0.08 \\ -0.10 & -0.46 & 1.04 \end{pmatrix}. \quad (10)$$

Correct reproduction is, therefore, obtained if the cyan element is formed from the red filter negative, itself reproduced to a gamma of 1.08, and masked by (*a*) a positive from the green filter negative of gamma 0.15 and (*b*) a positive from the blue filter negative of gamma 0.01. The other elements of (c^{-1}) have a similar significance.

Correct reproduction with a set of primaries characterized by (9) would, therefore, require three negatives and six positive masks. Most attempts at high quality of reproduction, however, seek an approximate rendition by the use of fewer masks. In such cases some of the elements of (m) are taken arbitrarily as zero. When this is done, not even neutral grays will be reproduced as neutral tones unless the densities, x_1, y_2, z_3, which are yielded by (4) are suitably modified. In practice this is done by adjusting empirically the concentration of the dyes which are to be used, in such a manner as to reproduce grays correctly. A theoretical estimate of the results to be achieved by such partial masking can be obtained by operating on D_0 with a "neutralizing" matrix (n) such that the transformation

$$D = (c)(n)D_0 = (c)(n)(m)d \quad (11)$$

shall yield correct grays. The criterion for this is that when $d = (1, 1, 1)$, it shall also be true that $D = (1, 1, 1)$. This condition can always be satisfied by a matrix (n) in which all except the elements on the principal diagonal are zero, i.e., by a matrix of the form

$$(n) = \begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{pmatrix}, \quad (12)$$

To determine the values of n_1, n_2 and n_3, we write down the form which the component equations of (11) take when both D and d are given the value (1, 1, 1). When a matrix such as (m) operates on the vector (1, 1, 1), the result is a vector of which the three components are, respectively, the sum of the elements in the three rows of the matrix. The operation of (n) on this resultant vector merely multiplies the three components by n_1, n_2 and n_3, respectively. If (11) is multiplied through by (c^{-1}), this would cancel the (c) on the right-hand side and leave $(c^{-1})D$ on the left. When D is given the value (1, 1, 1) the left-hand side becomes a vector which has as its components the sum of the three rows of (c^{-1}). The three component equations to which (11) thus reduces yield the following values for n_1, n_2 and n_3:

$$n_1 = \frac{c^{-1}_{1x} + c^{-1}_{2x} + c^{-1}_{3x}}{m_{1x} + m_{1y} + m_{1z}}, \quad n_2 = \frac{c^{-1}_{1y} + c^{-1}_{2y} + c^{-1}_{3y}}{m_{2x} + m_{2y} + m_{2z}},$$
$$n_3 = \frac{c^{-1}_{1z} + c^{-1}_{2z} + c^{-1}_{3z}}{m_{3x} + m_{3y} + m_{3z}}. \quad (13)$$

n_1, n_2 and n_3 are, therefore, simply the ratios of corresponding rows of (c^{-1}) and (m). Once (n) has been determined, it is most conveniently combined with (c) to characterize the reproduction which results from the hypothetical masking which is characterized by (m). $(c)(n)$ is, in fact, the matrix which is obtained from (c) by multiplying the first column of (c) by n_1, the second by n_2, and the third by n_3. This matrix will be referred to as the neutralized color matrix, and will be represented by the symbol (c_n).

$$(c_n) = (c)(n) = \begin{pmatrix} n_1 c_{1x} & n_2 c_{2x} & n_3 c_{3x} \\ n_1 c_{1y} & n_2 c_{2y} & n_3 c_{3y} \\ n_1 c_{1z} & n_2 c_{2z} & n_3 c_{3z} \end{pmatrix}. \quad (14)$$

Two special cases are of interest, that of no masking, in which (m) is the identical matrix, and that of ideal masking, in which $(m) = (c^{-1})$. In the first case of conventional reproduction, which makes use only of the single complementary filter negative for the determination of each constituent, (c_n) is obtained from (c) by multiplying each column by the sum of the elements in the corresponding row of (c^{-1}). For the typical case of (9),

$$(c_n) = \begin{pmatrix} 0.92 & 0.07 & 0.01 \\ 0.48 & 0.49 & 0.04 \\ 0.29 & 0.22 & 0.48 \end{pmatrix}. \quad (15)$$

The elements of the main diagonal are the gammas to which each constituent must be reproduced. The character of the reproduction

when dye concentrations are adjusted to give correct rendition of grays is given by the equation

$$D=(c_n)d. \tag{16}$$

In the other case of particular interest, that in which $(m)=(c^{-1})$, $n_1=n_2=n_3=1$, $(n)=I$, and $D=d$.

The great value of this method of approach to the problem of color reproduction is to be found in the light which it sheds on the effect of successive photographic reproductions such as are involved in the production of color prints or photoengravings from Kodachrome transparencies. Kodachrome, like all automatic color processes, seeks an approximate rendition, neutralized to reproduce grays correctly. The character of the reproduction which it achieves is, therefore, determined by a neutralized color matrix, which we will denote by (k_n) to distinguish it from (c_n), the matrix appropriate to the process which is to be used to reproduce the Kodachrome. The over-all reproduction which results from the two successive operations is characterized by the matrix product (c_nk_n) of the two individual matrices.

$$D=(c_nk_n)d. \tag{17}$$

If the same uncorrected primaries which are characterized by (c) were used in Kodachrome, the over-all reproduction would be described by the equation

$$D=(c_n^2)d. \tag{18}$$

The elements of (c_n^2) for the special case already considered, (9), are

$$(c_n^2)=\begin{pmatrix}0.88 & 0.10 & 0.02\\ 0.69 & 0.28 & 0.04\\ 0.51 & 0.24 & 0.24\end{pmatrix}. \tag{19}$$

The deterioration which results from two such consecutive operations is apparent from the increased magnitude of the nondiagonal elements.

For a method such as Kodachrome, which is processed commercially with a substantial degree of variability, it is important to be able to check the value of (k_n) from roll to roll, and for this purpose it is most convenient to characterize a color, not in terms of the three reflection densities d, but in terms of the densities d_0 of three chosen primaries which yield the same color. If we use for this characterization the set (c),

$$d=(c)d_0$$

or

$$d_0=(c^{-1})d. \tag{20}$$

The inverse matrix (c^{-1}) serves to analyze the color into its constituent primaries. If, now, small patches of these three primaries of unit density are photographed with Kodachrome, the way in which they are reproduced will be described by the three densities of the Kodachrome reproduction of each patch, i.e., by a new matrix which we will denote by (p).

$$(p)=\begin{pmatrix}p_{1x} & p_{2x} & p_{3x}\\ p_{1y} & p_{2y} & p_{3y}\\ p_{1z} & p_{2z} & p_{3z}\end{pmatrix}. \tag{21}$$

Here p_{1x}, p_{1y}, and p_{1z} are the densities of the reproduction of the cyan patch, p_{2x}, p_{2y}, p_{2z}, those of the magenta patch, and p_{3x}, p_{3y}, p_{3z}, those of the yellow patch. The over-all Kodachrome reproduction will be obtained by the successive operation of (c^{-1}), which analyzes the subject into its primary constituents, and the matrix (p) which indicates the effect of the reproduction of these primaries, i.e.,

$$D=(k_n)d=(p)(c^{-1})d. \tag{22}$$

If such primary color patches are included on each roll of Kodachrome (or as a part of each sheet of cut film), it is a relatively simple matter to make the necessary density measurements for determining (p) and therefore (k_n) indirectly. The values of (p) and (k_n) from one recent roll of Kodachrome are as follows:

$$(p)=\begin{pmatrix}1 & 0.24 & 0.09\\ 0.62 & 1 & 0.27\\ 0.34 & 0.57 & 1\end{pmatrix},$$

$$(k_n)=\begin{pmatrix}0.95 & 0.07 & 0.06\\ 0 & 0.99 & 0.19\\ -0.04 & 0.12 & 0.99\end{pmatrix}. \tag{23}$$

The remarkably low values of the nondiagonal elements of (k_n) indicate that the Eastman Kodak Company is utilizing the possibility which their new processing method provides for quite effective automatic masking. The fact that the

matrix (k_n) obviously will not reproduce neutral grays correctly is in accord with the fact that they were not well-rendered in this particular film.

The question now arises as to the nature of the masking which would be required in the reproduction to compensate for both the defects in the reproducing primaries (c) and the residual errors in the Kodachrome. It is not difficult to persuade oneself that, just as the correct masking matrix for a single reproduction is (c^{-1}), the correct matrix for reproducing a Kodachrome is (p^{-1}). For the density of the Kodachrome, as we have seen, is $(p)(c^{-1})d$. If we operate on this with (p^{-1}) as the masking matrix, we get for D_0, the densities in which the ultimate primaries are to be printed,

$$D_0 = (p^{-1})(p)(c^{-1})d = (c^{-1})d.$$

Hence

$$D = (c)D_0 = (c)(c^{-1})d = d$$

and correct reproduction results. The entire succession of operations may be expressed in the single equation,

$$D = (c)(p^{-1})(p)(c^{-1})d = d. \qquad (24)$$

It is a familiar fact that since the original Kodachrome has carried out the analysis of the colors in the subject, it is not necessary to use overlapping filters for analyzing the Kodachrome. Sharper cutting filters are, indeed, commonly used for making separation negatives from Kodachrome. The standard Wratten filters, 25, 58 and 47 were used in determining the elements of (p) which are recorded in (23). If the sharper cutting filters, 29, 61 and 49, had been used, and the masking matrix (p^{-1}) had been computed on this basis, its nondiagonal elements would have been somewhat smaller and the requisite masking a little less severe. The equation (24) would, however, still be correct. In case partial masking was resorted to, the same methods which were outlined for the single reproduction could be used to determine the neutralizing matrix, and so to compute the consequences of the proposed scheme. Although the reproduction of a Kodachrome transparency has been taken as a typical and perhaps the most important example of two successive reproductions, the procedure is naturally of much more general application.

A closely related problem to which these methods apply is that of preparing a reproduction in such a manner as to compensate for the alterations to be anticipated in a subsequent reproduction. If (p) controls the second reproduction in the same manner that it controlled the Kodachrome reproduction in the last problem, it is again true that the correct masking matrix is (p^{-1}). The proof is again given in substantially the same fashion. Starting now with the subject density d, this is operated on by the masking matrix (p^{-1}) and then by the color matrix (c). The second process, characterized by the matrix $(p)(c^{-1})$, must now operate on this resultant density to produce the final reproduction. Or

$$D = (p)(c^{-1})(c)(p^{-1})d = d. \qquad (25)$$

The question must now be faced as to the extent to which the above results are vitiated by the non-additivity of densities. The most important cause of non-additivity is scattering, the same agency which sets an upper limit to the reflection density of a photographic paper. Three different problems are presented by three different types of reproduction in which scattering plays a progressively increasing role. In photoengravings in which the elements of the mosaic pattern do not overlap, we have an example of complete additivity. In a photographic print which is formed by dye-imbibition, the three primary impressions are interpenetrating, and occupy jointly the volume of the gelatin coating of the paper. In this case the effect of scattering is to reduce all densities, approximately in the same proportion. It can be compensated for by making all three impressions somewhat heavier than would be required by the diagonal elements of (15). This would in turn require that the gammas of all constituents, both negatives and masks, should be increased in the same proportion.

Trichrome carbro is an example of the third type, in which the three impressions are deposited in layers. In this sort of print the densities of the lower layers must be increased to a greater extent than those of the upper layers in order to compensate for scattering. When this has been done to the extent which is necessary in order to yield correct grays, the net effect is very similar to

that which is produced by masking the under layers with positives from the negatives which control the upper layers. This effect is, on the whole, advantageous, and is doubtless responsible in some measure for the well-known beauty of trichrome carbro prints. Although the significance of the diagonal elements of (15) is almost completely lost in such cases, the information which (10) gives regarding the ratios of the desirable gammas of the masks to those of the negatives which they are designed to modify, is still, to a large extent, valid.

JULY, 1941 J. O. S. A. VOLUME 31

Colors of Ceramic Tiles

PARRY MOON
Massachusetts Institute of Technology, Cambridge, Massachusetts
(Received April 22, 1941)

1. SPECTRAL CURVES

IN the lighting and decoration of rooms, there is often a need for data on the reflection factors and colors of materials; and as the engineering aspects of lighting become more thoroughly developed, this need for data manifests itself ever more strongly. A previous paper[1] considered optical reflection factors and trichromatic coefficients for acoustical materials, while the present paper is devoted exclusively to ceramic tiles.

Samples were obtained from several tile manufacturers in the United States. Each sample was tested by means of the Hardy photoelectric spectrophotometer at the Massachusetts Institute of Technology,[2] approximately normal incidence being employed as usual. The reflected light, including all the specular component, is collected by an integrating sphere. The resulting curves give $\rho(\lambda)/\rho_0(\lambda)$ as a function of wave-length, where $\rho(\lambda)$ = true spectral reflection factor of the sample at wave-length λ, and $\rho_0(\lambda)$ = true spectral reflection factor of the MgO comparison surface at the same wave-length.

Curves for various ceramic tiles are shown in Figs. 1 to 14. Figures 1 and 2 show the spectrophotometric curves for a number of tiles that are relatively nonselective. Some of the white tiles have values of ρ/ρ_0 as high as 0.90 at the longer wave-lengths, but the reflection from all the whites is seen to decrease rapidly in the blue and violet parts of the spectrum. No. 177 is a

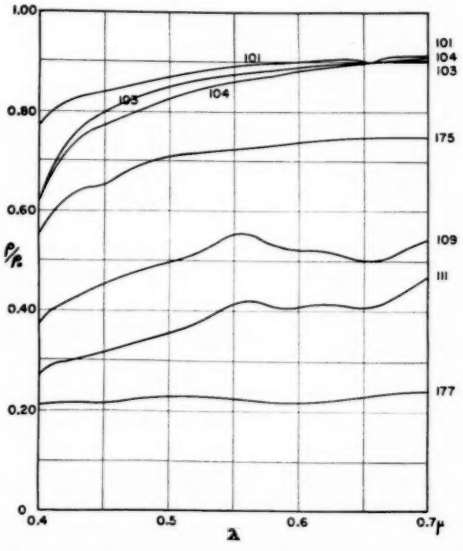

FIG. 1. Spectrophotometric curves for white and gray tiles. Numbers on the right margin refer to Table I.

[1] Parry Moon, "Optical Reflection Factors of Acoustical Materials," J. Opt. Soc. Am. **31**, 317 (1941).

[2] I wish to thank the staff of the Color Measurements Laboratory at Massachusetts Institute of Technology for the important part they played in this investigation. I am indebted to the National Youth Administration for assistance in the computation of the values of x, y, and ρ; and to the students who did this work: Messrs. E. D. Benson, F. Carroll, L. E. Duval, J. M. Fiore, J. Margolskee, J. P. Reilly, D. L. Shapiro, N. A. Teixeira, W. J. Thurston. It is a pleasure also to acknowledge the help of the various manufacturers who supplied samples of their products.

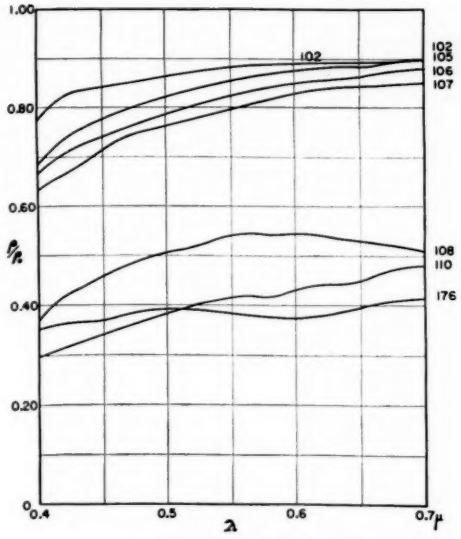

FIG. 2. White and gray tiles.

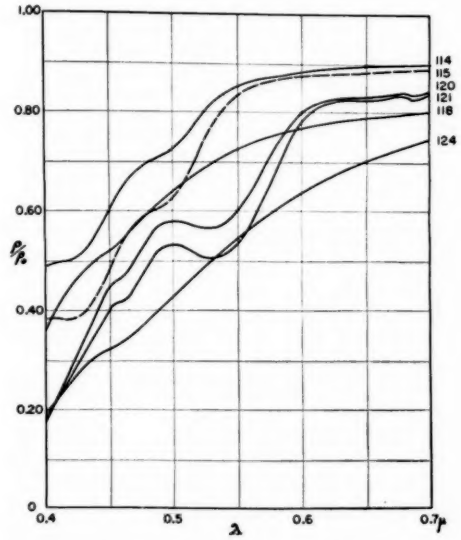

FIG. 3. Ivory, cream, and tan tiles.

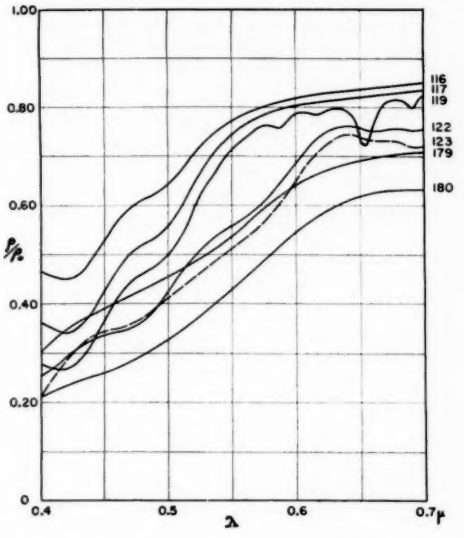

FIG. 4. Ivory, cream, and tan tiles.

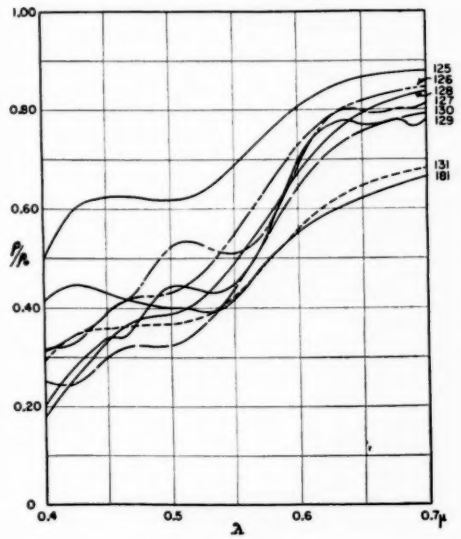

FIG. 5. Pink tiles.

neutral gray, while Nos. 109 and 111 are evidently somewhat greenish.

Figures 3 and 4 give results for a group of cream-colored tiles. The samples of Fig. 5 are low in the blue-green part of the spectrum and therefore appear pink, while those of Fig. 6 appear yellow because of their low reflection of blue light. The remaining curves show various greens, blues, and other colors.

2. TABULATED VALUES

Spectral curves, such as those given by Figs. 1 to 14, give a great deal of information on the character of the reflected light; but usually the

illuminating engineer desires the more compact specification given by the total reflection factor. Accordingly, the reflection factor was calculated for each of the spectrophotometric curves and for both Illuminants *A* and *C*. The former corresponds to the illumination of the samples with light from incandescent lamps, while the latter corresponds roughly to daylight illumination or to the use of daylight fluorescent lamps.

The reflection factors were obtained by the usual process of mechanical integration. For Illuminant *A*, some of the integrations were made by the Hardy machine, which uses 100 selected ordinates, and some were obtained from

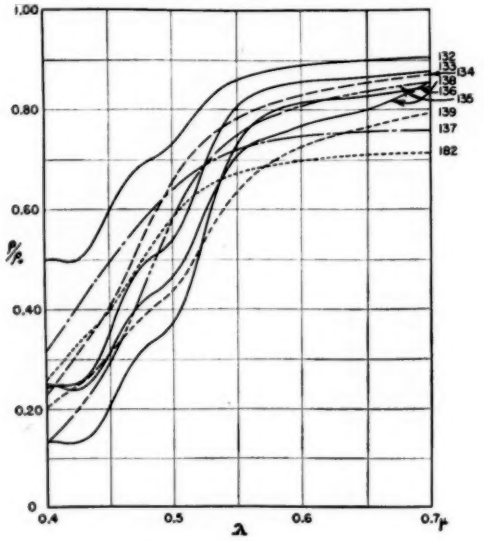

FIG. 6. Yellow tiles.

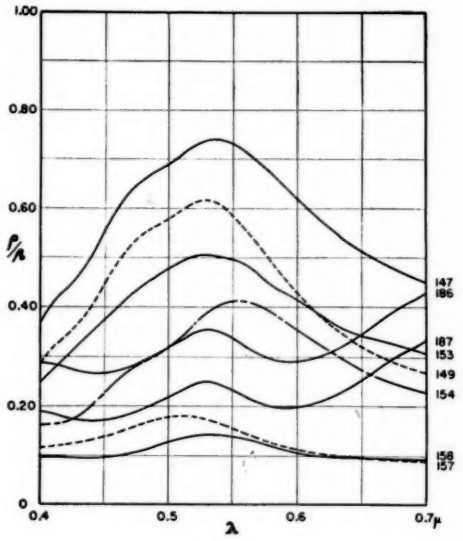

FIG. 8. Green tiles.

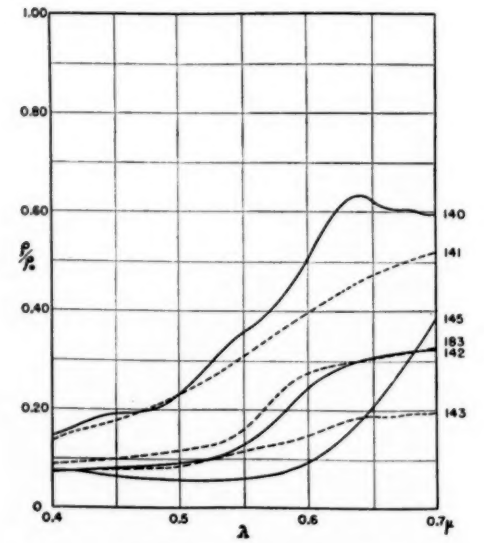

FIG. 7. Brown and red tiles.

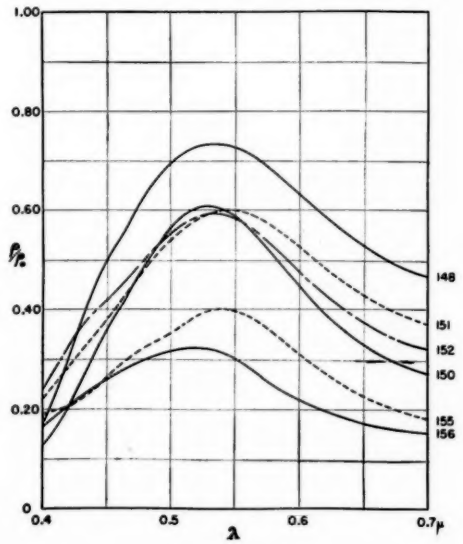

FIG. 9. Green tiles.

the spectrophotometric curves by the addition of 30 selected values. For Illuminant *C*, all values were obtained by use of 30 selected ordinates. The results were relative values (ρ/ρ_0) but were changed to absolute values by multiplication by $\rho_0=0.980$. These values of true reflection factor are given in Table I.

Also given in the table are the trichromatic coefficients x and y, which were computed from the curves by the usual method. The first column of Table I gives a designating number which agrees with the number on the curve sheets. The second column gives the manufacturer's specification, while the third column

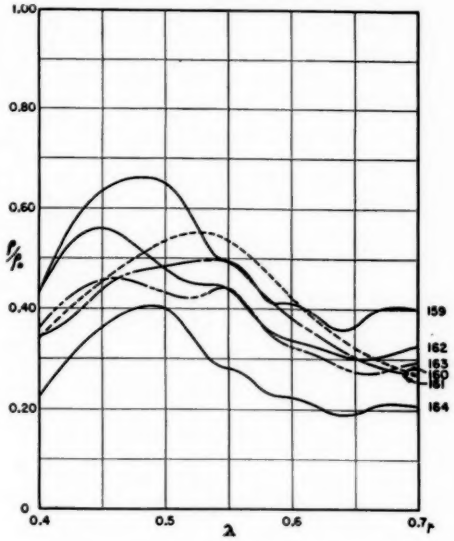

FIG. 10. Blue tiles.

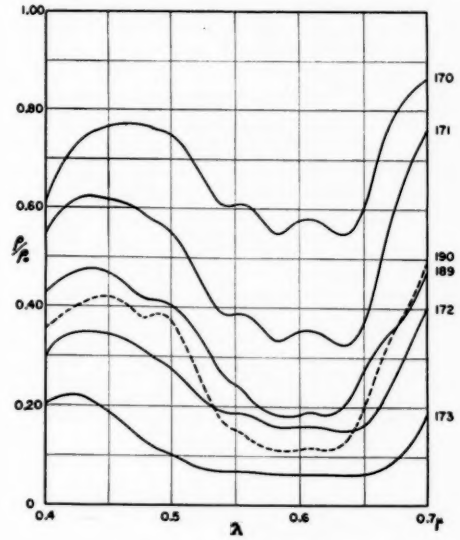

FIG. 12. Cobalt blue tiles.

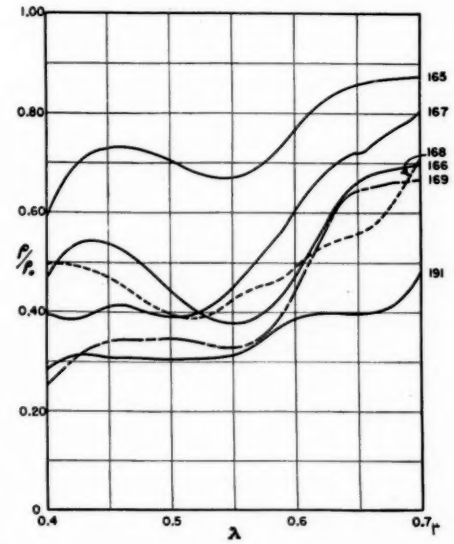

FIG. 11. Tiles reflecting both ends of the visible spectrum.

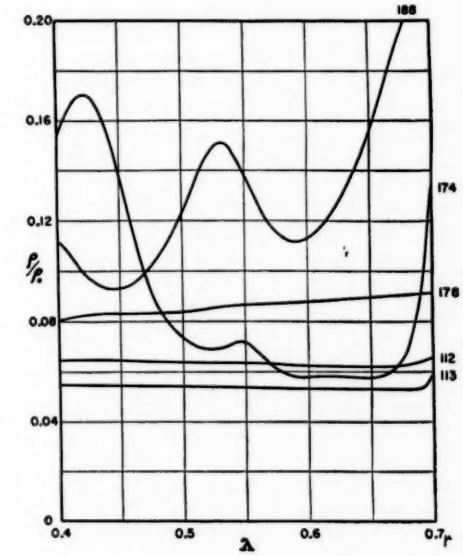

FIG. 13. Black and other dark tiles.

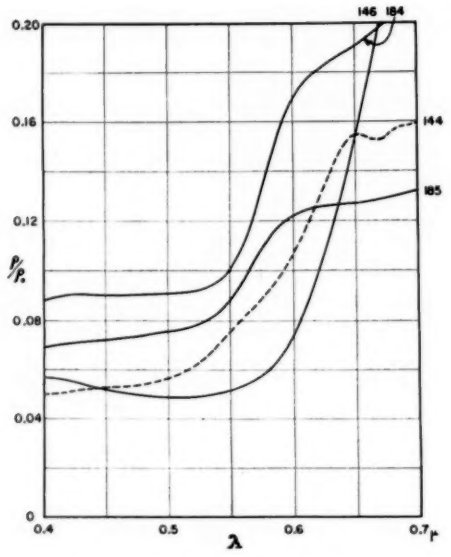

FIG. 14. Dark red and brown tiles.

designates the manufacturer. (See Table II of the paper on acoustical materials.) The fourth column gives some idea of the appearance of the material.

The computations, though easy to make, allow plenty of chance for error. Accordingly, three checks were made. All numerical work was done twice, generally by different computers. The integrated values were then checked roughly against the curves to eliminate the possibility of gross errors. Finally, the approximate values of ρ for daylight illumination were obtained visually and were compared with the values computed for Illuminant *C*.

It is hoped that the curves and tabulated values will be useful to those who are interested in lighting and decoration. The values of reflection factor allow the illuminating engineer to calculate the illumination in rooms having tiled floors or walls and to predetermine the

TABLE I. *Reflection factors and trichromatic coefficients.*

No.	Name	Mfgr.	Approx. Color	A			C		
				x	y	ρ	x	y	ρ
			Wall Tiles, Gray						
101	Suntile S-29	5	White	0.4524	0.4098	0.875	0.3157	0.3232	0.871
102	Mosaic No. 6028	21	White	0.4512	0.4095	0.868	0.3146	0.3219	0.864
103	Suntile S-8	5	White	0.4545	0.4114	0.863	0.3188	0.3280	0.857
104	Mosaic No. 409	21	White	0.4567	0.4114	0.850	0.3210	0.3292	0.843
105	No. 800, crystal white	23	White	0.4559	0.4111	0.844	0.3202	0.3278	0.837
106	No. 600, satin white	23	White	0.4573	0.4110	0.816	0.3214	0.3284	0.808
107	Heavy tile, No. 800 white glaze	23	White	0.4584	0.4108	0.793	0.3227	0.3297	0.784
108	Suntile S-27	5	Gray	0.4562	0.4150	0.525	0.3230	0.3343	0.520
109	No. 865, pearl gray	23	Lt. gray	0.4527	0.4178	0.518	0.3202	0.3370	0.517
110	Suntile S-4	5	Gray	0.4637	0.4135	0.411	0.3293	0.3378	0.404
111	No. 666, light olive gray	23	Gray	0.4622	0.4178	0.395	0.3307	0.3428	0.389
112	No. 699, satin black	23	Black	0.4449	0.4074	0.0620	0.3070	0.3148	0.0622
113	No. 899, mirror black	23	Black	0.4457	0.4074	0.0526	0.3078	0.3148	0.0527
			Wall Tiles, Cream						
114	Suntile S-32	5	Ivory	0.4689	0.4202	0.835	0.3400	0.3542	0.816
115	Suntile S-30	5	Ivory	0.4788	0.4259	0.813	0.3564	0.3718	0.785
116	No. 801, ivory	23	Ivory	0.4734	0.4197	0.765	0.3452	0.3566	0.745
117	No. 837, cream	23	Cream	0.4834	0.4245	0.737	0.3608	0.3733	0.709
118	No. 602, ivory	23	Ivory	0.4709	0.4185	0.722	0.3406	0.3520	0.705
119	No. 603, cream	23	Cream	0.4893	0.4261	0.704	0.3703	0.3824	0.675
120	Suntile S-9	5	Tan	0.4917	0.4107	0.675	0.3617	0.3560	0.641
121	Suntile S-10	5	Tan	0.5002	0.4064	0.635	0.3712	0.3544	0.594
122	No. 812, buff tan	23	Tan	0.4991	0.4152	0.597	0.3738	0.3681	0.562
123	Suntile S-31	5	Tan	0.4994	0.4100	0.559	0.3712	0.3575	0.525
124	Suntile S-25	5	Tan	0.4929	0.4184	0.568	0.3686	0.3694	0.540
			Wall Tiles, Pink						
125	Suntile S-2	5	Pink	0.4748	0.4045	0.724	0.3369	0.3274	0.698
126	No. 806, peach bloom	23	Pink	0.5006	0.4073	0.614	0.3718	0.3525	0.574
127	No. 605, peach	23	Pink	0.4945	0.4054	0.588	0.3602	0.3493	0.558
128	Mosaic No. 217	21	Pink	0.5070	0.4064	0.562	0.3803	0.3572	0.521
129	Mosaic No. 7046	21	Pink	0.5088	0.4022	0.555	0.3782	0.3525	0.514
130	No. 807, dark peach bloom	23	Pink	0.5150	0.4027	0.517	0.3887	0.3555	0.474
131	No. 615, deep rose pink	23	D. Pink	0.4929	0.4016	0.477	0.3568	0.3352	0.451
			Wall Tiles, Yellow						
132	Suntile S-18	5	Lt. Yellow	0.4692	0.4210	0.845	0.3410	0.3564	0.827
133	Suntile S-19	5	Yellow	0.4895	0.4312	0.786	0.3750	0.3938	0.753
134	No. 833, primrose yellow	23	Yellow	0.4808	0.4290	0.779	0.3602	0.3831	0.755
135	No. 835, jonquil yellow	23	Yellow	0.4915	0.4342	0.746	0.3786	0.4054	0.716
136	Suntile S-20	5	Yellow	0.5049	0.4332	0.717	0.4089	0.4206	0.669
137	Suntile S-13	5	Yellow	0.4690	0.4216	0.710	0.3403	0.3578	0.697
138	Mosaic No. 212	21	Yellow	0.4936	0.4290	0.697	0.3793	0.3919	0.663
139	Mosaic No. 463F	21	Yellow	0.4962	0.4249	0.646	0.3787	0.3848	0.611

TABLE I.—*Continued.*

No.	Name	Mfgr.	Approx. Color	A x	A y	A ρ	C x	C y	C ρ
			Wall Tiles, Brown						
140	No. 814, cinnamon tan	23	Tan	0.5247	0.4075	0.415	0.4049	0.3719	0.371
141	Suntile S-1	5	Lt. Brown	0.5074	0.4121	0.340	0.3842	0.3689	0.317
142	Suntile S-3	5	Brown	0.5421	0.3935	0.180	0.4244	0.3568	0.157
143	No. 694, chocolate	23	Dk. Brown	0.5050	0.4066	0.129	0.3763	0.3522	0.119
144	No. 895, chestnut	23	Dk. Brown	0.5198	0.3978	0.0904	0.3900	0.3489	0.0822
145	Suntile S-16	5	Dk. Red	0.5340	0.3696	0.0827	0.3809	0.3091	0.0736
146	No. 896, dubonnet	23	Dk. Red	0.5232	0.3764	0.0670	0.3748	0.3132	0.0602
			Wall Tiles, Green						
147	Suntile S-28	5	Lt. Green	0.4346	0.4298	0.652	0.3085	0.3490	0.667
148	Suntile S-15	5	Lt. Green	0.4412	0.4343	0.657	0.3176	0.3633	0.669
149	Suntile S-23	5	Lt. Green	0.4146	0.4376	0.494	0.2921	0.3495	0.518
150	Suntile S-6	5	Lt. Green	0.4263	0.4473	0.498	0.3081	0.3772	0.518
151	No. 825, opal green	23	Green	0.4455	0.4364	0.541	0.3228	0.3680	0.548
152	Mosaic No. 196	21	Lt. Green	0.4310	0.4355	0.511	0.3079	0.3561	0.526
153	Mosaic No. 443	21	Lt. Green	0.4328	0.4315	0.441	0.3072	0.3499	0.451
154	No. 618, light jade green	23	Green	0.4513	0.4426	0.361	0.3336	0.3810	0.363
155	No. 623, ming jade	23	Green	0.4279	0.4413	0.339	0.3078	0.3629	0.349
156	Suntile S-24	5	Green	0.4094	0.4309	0.255	0.2850	0.3359	0.270
157	Suntile S-17	5	Dk. Green	0.4063	0.4284	0.133	0.2806	0.3326	0.142
158	No. 859, cossack dark green	23	Dk. Green	0.4289	0.4382	0.120	0.3055	0.3545	0.123
			Wall Tiles, Blue						
159	Mosaic No. 451	21	Lt. Blue	0.4028	0.4029	0.455	0.2692	0.2928	0.483
160	No. 821, pale nile blue	23	Lt. Blue	0.4189	0.4295	0.465	0.2924	0.3396	0.489
161	No. 632, nile blue	23	Blue	0.4183	0.4254	0.423	0.2910	0.3298	0.440
162	Mosaic No. 220	21	Lt. Blue	0.4033	0.4012	0.381	0.2701	0.2864	0.403
163	No. 852, columbia blue	23	Blue	0.4109	0.4142	0.369	0.2804	0.3075	0.386
164	Suntile S-22	5	Blue	0.3906	0.4069	0.257	0.2624	0.2938	0.277
			Wall Tiles, Purple						
165	Suntile S-11	5	Lt. Orchid	0.4621	0.3977	0.710	0.3190	0.3103	0.696
166	Suntile S-21	5	Orchid	0.4670	0.3782	0.436	0.3126	0.2818	0.422
167	No. 846, corallin	23	Dk. Pink	0.4942	0.3971	0.514	0.3556	0.3275	0.483
168	No. 841, light lilac	23	Orchid	0.4677	0.3939	0.448	0.3236	0.3032	0.434
169	Suntile S-26	5	Orchid	0.4948	0.3893	0.394	0.3492	0.3185	0.371
170	Suntile S-7	5	Lt. Blue	0.4270	0.3967	0.593	0.2853	0.2933	0.609
171	Suntile S-5	5	Blue	0.4070	0.3821	0.378	0.2640	0.2654	0.397
172	Suntile S-12	5	Blue	0.3925	0.3725	0.179	0.2523	0.2479	0.191
173	Suntile S-14	5	Dk. Purple	0.3773	0.3400	0.0687	0.2387	0.2007	0.0728
174	No. 893, royal blue	23	Dk. Blue	0.3940	0.3642	0.0631	0.2544	0.2278	0.0660
			Floor Tiles						
175	Imper-Vit, white	23	White	0.4546	0.4114	0.715	0.3186	0.3279	0.710
176	Imper-Vit, light gray	23	Lt. Gray	0.4478	0.4090	0.373	0.3105	0.3204	0.374
177	Imper-Vit, dark gray	23	Dk. Gray	0.4479	0.4092	0.219	0.3119	0.3223	0.219
178	Imper-Vit, black	23	Black	0.4526	0.4084	0.0853	0.3149	0.3209	0.0850
179	Imper-Vit, cream	23	Cream	0.4856	0.4120	0.568	0.3552	0.3497	0.542
180	Imper-Vit, buff	23	Buff	0.5019	0.4127	0.467	0.3765	0.3653	0.438
181	Imper-Vit, pink	23	Pink	0.4816	0.3958	0.476	0.3401	0.3154	0.453
182	Imper-Vit, canary	23	Yellow	0.4741	0.4262	0.663	0.3494	0.3713	0.647
183	Imper-Vit, salmon	23	Red	0.5281	0.4018	0.208	0.4083	0.3629	0.187
			Floor Tiles						
184	Imper-Vit, red	23	Red	0.5118	0.3928	0.130	0.3759	0.3321	0.119
185	Imper-Vit, chocolate	23	Brown	0.4945	0.4044	0.101	0.3620	0.3390	0.0946
186	Imper-Vit, light sage	23	Green	0.4484	0.4192	0.312	0.3144	0.3387	0.313
187	Imper-Vit, dark sage	23	Green	0.4527	0.4214	0.214	0.3195	0.3462	0.214
188	Imper-Vit, green	23	Dk. Green	0.4562	0.4247	0.126	0.3239	0.3555	0.126
189	Imper-Vit, light blue	23	Lt. Blue	0.3810	0.3718	0.232	0.2422	0.2468	0.253
190	Imper-Vit, dark blue	23	Dk. Blue	0.3528	0.3476	0.160	0.2194	0.2181	0.180
191	Imper-Vit, orchid	23	Orchid	0.4718	0.4012	0.340	0.3314	0.3200	0.329

visual effect of changing from one kind of tile to another kind.

Also, since ceramic tiles are very permanent in reflecting properties and are carefully controlled for uniformity in manufacture, the present tabulation may allow tile samples to be used as rough comparison standards for judging the color and reflection factor of other materials. Munsell notation may be used, if desired, by comparing the x, y, ρ values of Table I with x, y, Y for the Munsell system, as given in a recent issue of this journal.[3,4]

[3] J. E. Tyler and A. C. Hardy, "An Analysis of the Original Munsell Color System," J. Opt. Soc. Am. **30**, 587 (1940).

[4] J. J. Glenn and J. T. Killian, "Trichromatic Analysis of the Munsell Book of Color," J. Opt. Soc. Am. **30**, 609 (1940).

A New Calculus for the Treatment of Optical Systems

I. Description and Discussion of the Calculus

R. CLARK JONES*
Research Laboratory, Polaroid Corporation, Cambridge, Massachusetts, and Research Laboratory of Physics, Harvard University, Cambridge, Massachusetts

(Received April 23, 1941)

The effect of a plate of anisotropic material, such as a crystal, on a collimated beam of polarized light may always be represented mathematically as a linear transformation of the components of the electric vector of the light. The effect of a retardation plate, of an anisotropic absorber (plate of tourmaline; Polaroid sheeting), or of a crystal or solution possessing optical activity, may therefore be represented as a matrix which operates on the electric vector of the incident light. Since a plane wave of light is characterized by the phases and amplitudes of the two transverse components of the electric vector, the matrices involved are two-by-two matrices, with matrix elements which are in general complex. A general theory of optical systems containing plates of the type mentioned is developed from this point of view.

INTRODUCTION

THE type of optical system with which this paper deals is not the more familiar type involving lenses, prisms, etc., but is rather the type which is composed of retardation plates, partial polarizers, and plates possessing the ability to rotate the plane of polarization. We shall therefore be concerned not with the directions of rays of light, but with the state of polarization and the intensity of the light as it passes through the optical system.

An example of a partial polarizer is a plate of tourmaline, or a sheet of light-polarizing film such as those sold under the trade-name "Polaroid." [1] For light incident perpendicularly on the plate, a partial polarizer may be characterized by two mutually perpendicular axes, with each of which is associated one of the two principal absorption coefficients. The retardation plate, or wave plate, is too well known to require description. In this paper we shall refer to a plate having the ability to rotate the plane of polarization as a *rotator*. A rotator is exemplified by a plate of quartz viewed along the optic axis, or by solutions of optically active molecules. We shall refer to the individual plates which make up the complete optical system as the elements of the system, or as the optical elements.

In the present paper it will be assumed that the light is always incident normally on the elements of the optical system.

The present paper is devoted to a description of the new calculus. When light passes through any of the three fundamental types of optical elements, the state of polarization and sometimes the intensity of the light will be changed. We shall find it possible to represent the effect of any optical element on the light as a linear operator acting upon the electric vector of the light wave. The operator is expressed in the convenient form of a two-by-two matrix, whose four matrix elements are in the general complex. From the associative property of matrices, it will then follow that the complete optical system may also be represented by a two-by-two complex matrix.

In Part II, the calculus is used to prove three theorems relating to optical systems of the type considered. The two theorems relating to systems containing partial polarizers are new, so far as the writer knows, and should be of some practical importance because of the recent development of a readily available and high quality polarizer in the form of Polaroid sheeting.

In Part III, the theory is applied to a rigorous treatment of the Sohncke theory of optical activity.

* National Scholar, Harvard University.

[1] Registered trade-mark of the Polaroid Corporation. For a description of Polaroid sheeting, see Martin Grabau, J. Opt. Soc. Am. **27**, 420 (1937).

Conventions and Notation

Consider a right-handed rectangular coordinate system, x, y and z. The elements of the optical system are considered to be arranged along the z axis, the z axis being perpendicular to the plane of the plates. The optical system will always be described as viewed from a point which is further out along the positive z axis than any of the elements; with this convention, the x and y axes have the relative position of the usual xy plane: the positive x axis may be superposed upon the positive y axis by rotating the x axis 90° counterclockwise.

Furthermore, let the two principal axes of the ith retardation plate or partial polarizer be indicated by x_i' and y_i'. We assume that the positive x_i' and y_i' axes have the same relative orientation as the positive x and y axes. We may now define the orientation of the ith element by stating the angle ω_i measured counterclockwise from the positive x axis to the positive x_i axis.

The light will be represented as plane waves progressing in either direction along the z axis. The state of polarization may be completely defined by stating the amplitudes and phases of the x and y components of the electric vector of the light wave. At any fixed point along the z axis, the components may be written in the usual complex form

$$\begin{aligned} E_x &= A_x \exp\left[i(\epsilon_x + 2\pi\nu t)\right], \\ E_y &= A_y \exp\left[i(\epsilon_y + 2\pi\nu t)\right], \end{aligned} \tag{1}$$

where A_x and A_y, ϵ_x and ϵ_y are real. If $\epsilon_x - \epsilon_y$ is an integral multiple of π, the light is plane polarized; otherwise, elliptically polarized.

Now let us consider the change in the character of a light wave as it passes through a retardation plate or a partial polarizer. As a matter of fact, in order to avoid giving a separate treatment of the two cases, we shall suppose that both the indices of refraction and the absorption coefficients are different along the two principal axes (x' and y') of the plate. If we know the x' and y' components ($E_{x'0}$ and $E_{y'0}$) of the electric vector as the light enters the plate, then the corresponding components of the light as it emerges from the other side of the plate are

$$\begin{aligned} E_{x'1} &= E_{x'0} \exp\left[-i(2\pi d/\lambda)(n_{x'} - ik_{x'})\right] \\ &= N_{x'} E_{x'0}, \\ E_{y'1} &= E_{y'0} \exp\left[-i(2\pi d/\lambda)(n_{y'} - ik_{y'})\right] \\ &= N_{y'} E_{y'0}. \end{aligned} \tag{2}$$

In these expressions, d is the thickness of the plate, λ is the wave-length of the light in vacuum, the n's are the principal indices of refraction, and the k's are the principal extinction coefficients. The extinction coefficient is a particular type of amplitude absorption coefficient. The N's are merely abbreviations for the exponential expressions. In the case of a retardation plate, the k's are the same and the n's different, whereas in the case of a partial polarizer, the n's are the same and the k's different. In practice, of course, it is difficult to secure a partial polarizer which is not also birefringent.

In general, however, we are interested in knowing the change in the x and y components of the light wave, rather than the change in the x' and y' components. If ω is the angle measured counterclockwise from the positive x axis to the positive x' axis, we have the relations

$$\begin{aligned} E_{x'} &= E_x \cos\omega + E_y \sin\omega, \\ E_{y'} &= -E_x \sin\omega + E_y \cos\omega. \end{aligned} \tag{3}$$

The elimination of the x' and y' components in (2) and (3) yields the result

$$\begin{aligned} E_{x1} &= (N_{x'}\cos^2\omega + N_{y'}\sin^2\omega)E_{x0} \\ &\quad + (N_{x'} - N_{y'})\sin\omega\cos\omega E_{y0}, \\ E_{y1} &= (N_{x'} - N_{y'})\sin\omega\cos\omega E_{x0} \\ &\quad + (N_{x'}\sin^2\omega + N_{y'}\cos^2\omega)E_{y0}. \end{aligned} \tag{4}$$

The equations (4) give the important relation between the x and y components of the entering and emergent light.

The Matrix Notation

The relations (4) may be put in a convenient and simple form by the use of a matrix notation.[2]

[2] In Parts I and II we shall need only the algebraic properties of matrices, as they are presented on pp. 348–352 of E. C. Kemble, *Fundamental Principles of Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1937). In Part III, we shall find it necessary to use also the transformation properties of matrices; see the reference just given, or V. Rojansky, *Introductory Quantum Mechanics* (Prentice-Hall, New York, 1938), pp. 285–340.

Let $\mathbf{M}$ be the two-by-two matrix*

$$\mathbf{M} \equiv \begin{pmatrix} m_1 & m_4 \\ m_3 & m_2 \end{pmatrix}, \tag{5}$$

where

$$\begin{aligned} m_1 &= N_{x'} \cos^2 \omega + N_{y'} \sin^2 \omega, \\ m_2 &= N_{x'} \sin^2 \omega + N_{y'} \cos^2 \omega, \\ m_3 &= m_4 = (N_{x'} - N_{y'}) \sin \omega \cos \omega. \end{aligned} \tag{6}$$

Furthermore, let $\mathcal{E}_0$ and $\mathcal{E}_1$ be the one-column vectors

$$\mathcal{E}_0 \equiv \begin{pmatrix} E_{x0} \\ E_{y0} \end{pmatrix}; \quad \mathcal{E}_1 \equiv \begin{pmatrix} E_{x1} \\ E_{y1} \end{pmatrix}. \tag{7}$$

The relations (4) are now equivalent to the vector equation

$$\mathcal{E}_1 = \mathbf{M}\mathcal{E}_0. \tag{8}$$

The matrix $\mathbf{M}$ may itself be written in a simple form. Let $\mathbf{S}(\omega)$ be the rotation matrix

$$\mathbf{S}(\omega) \equiv \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix} \tag{9}$$

and let

$$\mathbf{N} \equiv \begin{pmatrix} N_{x'} & 0 \\ 0 & N_{y'} \end{pmatrix}. \tag{10}$$

We then have

$$\mathbf{M} = \mathbf{S}(\omega)\mathbf{N}\mathbf{S}(-\omega) \tag{11}$$

or

$$\mathcal{E}_1 = \mathbf{S}(\omega)\mathbf{N}\mathbf{S}(-\omega)\mathcal{E}_0. \tag{12}$$

We have thus been able to represent the effect of a retardation plate or a partial polarizer as a matrix operator which operates upon the vector describing the intensity and polarization of the entering light. The matrix operator has been factored into a product of two types of matrices, the first of which, $\mathbf{N}$, describes the optical element in a way independent of its orientation, and the second of which, $\mathbf{S}$, describes the orientation.

Multi-Element Systems

Suppose now that we have a series of n optical elements, represented by the n matrices $\mathbf{N}_1$, $\mathbf{N}_2$, $\cdots$, $\mathbf{N}_n$, with the respective orientations ω_1, ω_2, $\cdots$, ω_n, and suppose that the light passes through the elements in the order in which they are numbered. We then have

$$\begin{aligned} \mathcal{E}_1 &= \mathbf{S}(\omega_1)\mathbf{N}_1\mathbf{S}(-\omega_1)\mathcal{E}_0 = \mathbf{M}_1\mathcal{E}_0, \\ \mathcal{E}_0 &= \mathbf{M}_2\mathcal{E}_1, \\ &\cdots\cdots\cdots \\ \mathcal{E}_n &= \mathbf{M}_n\mathcal{E}_{n-1}, \end{aligned} \tag{13}$$

where $\mathcal{E}_i$ is the light vector as it emerges from the ith element. By substituting each of the Eqs. (13) in the one following it, we have at once the relation between $\mathcal{E}_n$ and $\mathcal{E}_0$

$$\begin{aligned} \mathcal{E}_n &= \mathbf{M}_n\mathbf{M}_{n-1}\cdots\mathbf{M}_2\mathbf{M}_1\mathcal{E}_0 \\ &= \mathbf{M}^{(n)}\mathcal{E}_0, \end{aligned} \tag{14}$$

where $\mathbf{M}^{(n)}$ is the two-by-two matrix

$$\mathbf{M}^{(n)} \equiv \mathbf{M}_n\mathbf{M}_{n-1}\cdots\mathbf{M}_2\mathbf{M}_1. \tag{15}$$

According to (6), each of the $\mathbf{M}_i$'s is a symmetrical matrix. In general, however, the product of two or more of the $\mathbf{M}$ matrices will be neither symmetrical nor anti-symmetrical. This criterion permits us to distinguish immediately between the matrices corresponding to a single element, and those corresponding to the superposition of two or more elements.

It is fruitful to examine the structure of $\mathbf{M}^{(n)}$ more closely. We have

$$\mathbf{M}^{(n)} = [\mathbf{S}(\omega_n)\mathbf{N}_n\mathbf{S}(-\omega_n)][\mathbf{S}(\omega_{n-1})\mathbf{N}_{n-1}\mathbf{S}(-\omega_{n-1})]\cdots[\mathbf{S}(\omega_2)\mathbf{N}_2\mathbf{S}(-\omega_2)][\mathbf{S}(\omega_1)\mathbf{N}_1\mathbf{S}(-\omega_1)]. \tag{16}$$

By the use of the easily proved relation:

$$\mathbf{S}(\omega_1+\omega_2) \equiv \mathbf{S}(\omega_1)\mathbf{S}(\omega_2), \tag{17}$$

the expression for $\mathbf{M}^{(n)}$ may be written in either of the following forms:

$$\mathbf{M}^{(n)} = \mathbf{S}(\omega_n)[\mathbf{N}_n\mathbf{S}(\omega_{n,\,n-1})\cdots\mathbf{N}_2\mathbf{S}(\omega_{2,\,1})\mathbf{N}_1\mathbf{S}(\omega_{1,\,n})]\mathbf{S}(-\omega_n), \tag{18}$$

$$= \mathbf{S}(\omega_1)[\mathbf{S}(\omega_{1,\,n})\mathbf{N}_n\mathbf{S}(\omega_{n,\,n-1})\cdots\mathbf{S}(\omega_{2,\,1})\mathbf{N}_1]\mathbf{S}(-\omega_1), \tag{19}$$

where

$$\omega_{i,\,j} \equiv \omega_j - \omega_i. \tag{20}$$

* We depart from the usual double-subscript notation for matrix elements solely for reasons of typographical convenience.

The matrices represented by the square brackets in Eqs. (18) and (19) depend only on the relative orientation of the elements of the optical system, and are independent of the orientation of the optical system as a whole. Thus, just as in the case of a single element, we have factored the operator representing the optical system into the product of two types of matrices, the first of which depends only on the nature of the optical elements and on their *relative* orientation, and the second of which describes only the orientation of the optical system as a whole with respect to the x, y axes.

So far we have excluded any consideration of rotators, which are plates possessing the ability to rotate the plane of polarization about the z axis. The formal change caused by the introduction of rotators into the system is very small, however. Suppose that a given rotator is able to rotate the plane of polarization through the angle $\tilde{\omega}$ in the counterclockwise direction. The matrix operator corresponding to the rotator is then simply $\mathbf{S}(\tilde{\omega})$. We shall not stop to prove this statement,* since its proof is elementary. If the rotator just considered is introduced between the ith and the $(i+1)$th element of the optical system considered in the last paragraph, the resulting optical system may still be represented in the form of Eqs. (18) and (19) if we redefine $\omega_{i+1,i}$ as

$$\omega_{i+1,i} \equiv \omega_i - \omega_{i+1} + \tilde{\omega}. \tag{21}$$

The generalization to the case in which several rotators are placed in the optical system is obvious. If several rotators are placed in juxtaposition, they are equivalent to a single rotator, according to (17).

There is an important physical distinction between the behavior of retardation plates and partial polarizers on one hand, and the behavior of rotators on the other. With the coordinate system which we are using, it makes no difference in what direction the light passes through a retardation plate or partial polarizer. With the type of rotator represented by a crystal or solution possessing optical activity, however, the rotation changes sign if the light passes through the plate in the reverse direction. In determining the operator corresponding to a rotator, therefore, it is necessary to know in which direction the light passes through the plate.

* More precisely, the matrix of a rotator is $e^{i\phi}\mathbf{S}(\tilde{\omega})$, where the phase factor takes into account the finite optical thickness of the plate. In this paper, however, rotators will be considered only in combination with retardation plates and partial polarizers, so that the phase factor may always be considered to be associated with the other elements of the optical system. For simplicity, we omit the phase factor.

Reversibility of the Optical System

It is, however, not necessary to make a recalculation if the light passes through the optical system in the opposite direction. Suppose that we have computed the matrix $\mathbf{M}^{(n)}$ for the case that the light passes through the system in the order in which the plates are numbered, and then suppose that the light is reversed so that it passes through the system in the opposite direction, passing first through the nth plate, and last through the first plate. In terms of the one-row vectors,

$$\bar{\varepsilon} \equiv (E_x\, E_y), \tag{22}$$

the relation between the vector $\bar{\varepsilon}_{\text{initial}}$ of the light entering the nth plate and the vector $\bar{\varepsilon}_{\text{final}}$ of the light emerging from the first plate is

$$\bar{\varepsilon}_{\text{final}} = \bar{\varepsilon}_{\text{initial}}\mathbf{M}^{(n)}, \tag{23}$$

where $\mathbf{M}^{(n)}$ is exactly the same matrix as that previously defined. This statement may be proved easily, first by showing that a relation of the form (23) holds for each element of the system, and then by eliminating the intermediate $\bar{\varepsilon}$ vectors.

The statement made in the last paragraph holds only if all of the rotators in the system are of the type represented by ordinary optical activity, and not of the type represented by the Faraday effect, in which case the direction of the rotation depends on the direction of the magnetic field, and not on the direction in which the light passes through the material. If any of the rotators are of the type represented by the Faraday effect, then the $\mathbf{M}^{(n)}$ which appears in (23) is not the same as the $\mathbf{M}^{(n)}$ which appears in (14), the difference being that the $\tilde{\omega}$'s appearing in the $\mathbf{S}$ matrices which represent the Faraday rotators must have opposite signs in the two $\mathbf{M}^{(n)}$'s.

Discussion of the Matrices

When the optical system does not contain any partial polarizers but consists entirely of rotators

and retardation plates, *all of the matrices are unitary.*‡ This fact may be proved by either of two methods.

First proof

The **N** matrices are all unitary, since their characteristic values are all of absolute value unity; the latter is a necessary and sufficient condition that a matrix be unitary. Furthermore, the **S** matrices are easily shown to be unitary, since the conjugate transpose of $\mathbf{S}(\omega)$ is $\mathbf{S}(-\omega)$, so that $\mathbf{S}^\dagger\mathbf{S}=1$. The general matrix $\mathbf{M}^{(n)}$ is the product of factors which are all unitary, and therefore $\mathbf{M}^{(n)}$ is unitary.

Second proof

This proof is not purely mathematical, but is based on the fact that an optical system composed only of rotators and retardation plates does not change the intensity of the light passing through it. The intensity of the light is proportional to the sum of the squares of the x and y components of the electric vector:

$$I \propto A_x{}^2+A_y{}^2=E^*{}_xE_x+E^*{}_yE_y=\bar{\varepsilon}^*\varepsilon=\bar{\varepsilon}\varepsilon^* \quad (24)$$

(cf. Eq. (1) for part of the notation). The intensity of the light is therefore proportional to the square of the length of the vector ε. Now a unitary matrix does not change the length of a vector, and conversely, if a matrix does not change the length of an arbitrary vector, then the matrix is unitary. Thus, since the optical system does not change the intensity of the light, the corresponding matrices must be unitary.

Conversely, if the first proof of the unitary character of the matrices is accepted, then the second may be regarded as a proof that an optical system composed only of retardation plates and rotators does not change the intensity of the light passing through it.

In nearly every case, we are not interested in the total phase change corresponding to the two axes of a retardation plate, but only in the phase difference of the two principal axes; we often refer to a retardation plate as a q wave plate, implying that the optical path difference for plane polarized light parallel to each of the two axes is q wave-lengths.

Since the two principal extinction coefficients of a retardation plate are both zero, we find by comparison of (2) with (10) that the matrix of a retardation plate whose axes are parallel to the x and y axes may always be written in the form $e^{i\phi}\mathbf{G}$, where

$$\mathbf{G}\equiv\begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix}. \quad (25)$$

If we are interested only in the phase difference, the phase factor $e^{i\phi}$ may be omitted, and we may consider **G** alone to be the diagonal representation of a wave plate.

If we choose to write the diagonal matrices of all the wave plates in the optical system in the form (25), then, since the determinant of an S matrix is unity, all of the matrices which occur as factors on the right-hand side of Eq. (16) will have determinants equal to unity, provided that the system contains only retardation plates and rotators. Then, since the determinant of a product of matrices is equal to the product of the determinants of the matrices, the determinant of $\mathbf{M}^{(n)}$ will have the value unity.

Our treatment of partial polarizers is similar. The diagonal form of the matrix corresponding to a partial polarizer may always be written in the form $e^{i\phi}\mathbf{P}$, where

$$\mathbf{P}\equiv\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \quad \begin{matrix} 0\leqslant p_1\leqslant 1 \\ 0\leqslant p_2\leqslant 1. \end{matrix} \quad (26)$$

Again, if we are not interested in the absolute phase of the two components of the light, but only in the relative phase, we may consider **P** alone to be the diagonal representation of a partial polarizer. If the plate is a perfect polarizer, that is to say, if the plate absorbs completely light parallel to one of its axes, then one of the two p's will be zero; in this case the determinant of **P** is zero. More generally, the determinant of **P** is real, with its value lying in the range zero to unity.

We thus find that if we use **G** to represent a retardation plate, and **P** to represent a partial polarizer, then the determinant of $\mathbf{M}^{(n)}$ for an optical system containing all three types of

‡ Let $\mathbf{M}^\dagger$ be the matrix obtained from **M** by first exchanging m_3 and m_4 and then replacing each of the matrix elements by its complex conjugate. The matrix **M** is unitary if $\mathbf{M}^\dagger\mathbf{M}=\mathbf{M}\mathbf{M}^\dagger=1$. The matrix $\mathbf{M}^\dagger$ is said to be the Hermitian adjoint, or the conjugate transpose of **M**.

elements will be real and non-negative, with its value in the range zero to unity.

It will be shown in Part II that the most general unitary matrix may be represented by an optical system consisting of only two elements, one a rotator and the other a retardation plate. Similarly, it will be shown that the general two-by-two complex matrix may be represented by an optical system containing not more than four elements, the only restriction being that the matrix may not be one which would increase the intensity of the light.

Elliptical Polarization

For convenience in reference, we state here the relations giving the shape and orientation of the ellipse representing the state of polarization of the light. Let the amplitude of the x and y components be A_x and A_y, and let $\delta=\epsilon_y-\epsilon_x$, as in Eq. (1). Further, let $\tan\theta$ equal the ratio of axes of the ellipse, and let ψ be the orientation of the major axis of the ellipse, measured counterclockwise from the x axis. The following relations determine θ and ψ:[3]

$$\tan 2\psi=\tan 2\alpha\cos\delta,$$
$$\cos 2\theta=\sin 2\alpha|\sin\delta|, \tag{27}$$

where

$$\tan\alpha=A_y/A_x,\quad 0\leqslant\alpha\leqslant\tfrac{1}{2}\pi. \tag{28}$$

When $\sin\delta$ is positive, the ellipse is described in the clockwise direction, and *vice versa*.

[3] Max Born, *Optik* (Springer, Berlin, 1933), p. 23.

JULY, 1941 J. O. S. A. VOLUME 31

A New Calculus for the Treatment of Optical Systems

II. Proof of Three General Equivalence Theorems

Henry Hurwitz, Jr., *Research Laboratory of Physics, Harvard University, Cambridge, Massachusetts,*

and

R. Clark Jones,* *Research Laboratory, Polaroid Corporation, Cambridge, Massachusetts, and Research Laboratory of Physics, Harvard University, Cambridge, Massachusetts*

(Received April 23, 1941)

The general theory developed in Part I is used to prove three equivalence theorems about optical systems of the type under discussion. We prove that any optical system which contains only retardation plates and rotators is optically equivalent to a system containing only two plates—one a retardation plate, and the other a rotator. We then prove an exactly analogous theorem for systems containing only partial polarizers and rotators. Finally, it is proved that the most general optical system which contains any number of all three types of plates is optically equivalent to a system containing at most four plates—two retardation plates, one partial polarizer, and one rotator.

Statement of the Theorems

THE power of the calculus described in Part I may be illustrated by using it to establish a few general theorems relating to optical systems of the type under consideration.

We shall prove the following three theorems:

I. For light of a given wave-length, an optical system containing any number of retardation plates and rotators is optically equivalent to a system containing only two elements—one a retardation plate, and the other a rotator.

II. For light of a given wave-length, an optical system containing any number of partial polarizers and rotators is optically equivalent to a system containing only two elements—one a partial polarizer and the other a rotator.

III. For light of a given wave-length, an optical system containing any number of re-

* National Scholar, Harvard University.

tardation plates, partial polarizers, and rotators is optically equivalent to a system containing four elements—two retardation plates, one partial polarizer, and one rotator. In a large and finite class of cases, the rotator is not necessary.

The method of proof of these theorems is essentially the same for each of them. We shall first determine the mathematical restrictions on the matrix representing the original optical system. Secondly, we shall write down the matrix corresponding to the substitute optical system. We shall then equate these two matrices. In each case, we shall find that the resulting equations are of such a form that they always admit a physically realizable solution for the parameters of the substitute optical system. The proof that the equations are always solvable will be obtained by actually solving the equations explicitly for the parameters of the substitute optical system. The proof will thus not only establish that the replacement is always possible, but it will also provide the explicit equations for computing the parameters of the substitute optical system corresponding to any given original optical system.

The first of the three theorems has been proved by H. Poincaré by an entirely different method (cf. below). So far as the writers have been able to learn, the remaining two theorems are new.

Proof of Theorem I

We saw in Part I that the matrix $\mathbf{M}^{(n)}$ representing any optical system composed of retardation plates and rotators was unitary. We shall demonstrate that any unitary matrix may be represented by a system consisting of one retardation plate and one rotator.

We shall write

$$\mathbf{M}^{(n)}=e^{i\alpha}\mathbf{U} \tag{1}$$

and we shall select the angle α so that the determinant of $\mathbf{U}$ is unity.

It is well known[1] that the general unitary matrix with two rows and columns with unit determinant may be written in the form:

$$\mathbf{U}\equiv\begin{pmatrix} e^{i\phi}\cos\theta & -e^{-i\psi}\sin\theta \\ e^{i\psi}\sin\theta & e^{-i\phi}\cos\theta \end{pmatrix}. \tag{2}$$

The real angles θ, ϕ and ψ may be determined from the matrix elements of $\mathbf{U}$ by the relations:

$$\begin{aligned} e^{2i\phi}&=u_1/u_2, \quad \sin\theta=e^{-i\psi}u_3, \\ e^{2i\psi}&=u_3/u_4, \quad \cos\theta=e^{-i\phi}u_4. \end{aligned} \tag{3}$$

For any given $\mathbf{M}^{(n)}$, it is thus possible to determine the values of α, θ, ϕ and ψ, although the determination is not unique; for example, if ϕ is a solution, then ϕ plus an integral multiple of π is also a solution.

The matrix of a retardation plate with the orientation ω is $\mathbf{S}(\omega)\mathbf{G}\mathbf{S}(-\omega)$, where $\mathbf{G}$ is defined by Eq. (25, I); the matrix of a rotator is $\mathbf{S}(\tilde{\omega})$. Whether the rotator is the first or second element of the system, the matrix of the combination will be of the form

$$\mathbf{V}\equiv\mathbf{S}(A)\mathbf{G}\mathbf{S}(B), \tag{4}$$

where the angles A and B are in general different.

If now for any values of θ, ϕ and ψ, we are always able to choose A, B and γ so that $\mathbf{V}=\mathbf{U}$, our theorem is proved. We multiply out explicitly the expression on the right of (4), so that the matrix elements of $\mathbf{V}$ appear as explicit functions of A, B and γ. We then set the four elements of $\mathbf{U}$ equal, respectively, to the four elements of $\mathbf{V}$, and after a little algebraic reduction, we find

$$\begin{aligned} \cos\phi\cos\theta&=\cos(A+B)\cos\gamma, \\ \sin\phi\cos\theta&=\cos(A-B)\sin\gamma, \\ \cos\psi\sin\theta&=\sin(A+B)\cos\gamma, \\ \sin\psi\sin\theta&=\sin(A-B)\sin\gamma \end{aligned} \tag{5}$$

as the conditions which must be satisfied in order that $\mathbf{V}=\mathbf{U}$.

From (5), we find as the explicit solution for A, B and γ:

$$\tan(A+B)=\frac{\cos\psi}{\cos\phi}\tan\theta=\frac{u_3-u_4}{u_1+u_2}, \tag{6}$$

[1] B. L. van der Waerden, *Die Gruppentheoretische Methode in der Quantenmechanik* (Springer, Berlin, 1932), p. 58.

$$\tan (A-B)=\frac{\sin \psi}{\sin \phi}\tan \theta=\frac{u_3+u_4}{u_1-u_2}, \tag{7}$$

$$\tan \gamma=\frac{\cos (A+B)}{\cos (A-B)}\tan \phi$$

$$=\frac{\sin (A+B)}{\sin (A-B)}\tan \psi. \tag{8}$$

It is evident that these equations always have a solution. The theorem is thus established. If we set

$$\mathbf{N}\equiv e^{i\alpha}\mathbf{G}, \tag{9}$$

then we have

$$\mathbf{M}^{(\alpha)}=\mathbf{S}(A)\mathbf{N}\mathbf{S}(B). \tag{10}$$

We may at once show also that any matrix in the form **V** may be expressed in the form **U**. From (5) we have as the explicit solution for θ, ϕ and ψ in terms of A, B and γ:

$$\tan \phi=\frac{\cos (A-B)}{\cos (A+B)}\tan \gamma, \tag{11}$$

$$\tan \psi=\frac{\sin (A-B)}{\sin (A+B)}\tan \gamma, \tag{12}$$

$$\tan \theta=\frac{\cos \phi}{\cos \psi}\tan (A+B)=\frac{\sin \phi}{\sin \psi}\tan (A-B). \tag{13}$$

It is also evident that these equations always have a solution. We have thus shown that **U** and **V** are mathematically equivalent representations of the general unitary matrix with unit determinant. In the proof of Theorem III, we shall find it convenient to pass freely from one representation to the other.

Poincaré's Proof of Theorem I

Poincaré[2] has given a geometrical proof of Theorem I which appears at first sight to be quite different from ours. Actually, however, the two proofs are very closely related, as we shall now show.

Poincaré expresses the ratio of the two complex components of the light in the form

$$E_y/E_x=(A_y/A_x)\exp\left[i(\epsilon_y-\epsilon_x)\right]=\xi+i\eta. \tag{14}$$

In this expression, ξ and η together define completely the shape and orientation of the ellipse representing the state of polarization of the light. Thus each point on the ξ, η plane corresponds to a state of polarization, and conversely, each state of polarization corresponds to a point on the ξ, η plane.

Each point on the plane is considered to be projected stereographically upon a sphere of unit diameter to which the plane is tangent at the origin of coordinates. There is thus established a one-to-one correspondence between points on the sphere and points on the plane.

If now the sphere is subjected to a rotation about any diameter, and if we consider that the points on the sphere rotate rigidly with the sphere, then it is clear that all of the corresponding points on the plane (except two) will suffer a change of position. Thus a rotation of the sphere always may be said to correspond to a change in the state of polarization, and conversely, any change in the state of polarization may be related to some rotation of the sphere. Poincaré shows that the rotation corresponding to the action of a wave plate with retardation $\Delta=2\gamma$ is a rotation of angle Δ about the diameter which is perpendicular to the plane; he also shows that the rotation which corresponds to a rotator of angle $\tilde{\omega}$ is a rotation of angle $2\tilde{\omega}$ about the diameter which is parallel to the ξ axis.

Once the facts stated in the last paragraph have been established, the proof of Theorem I follows immediately, since it is physically obvious that any rotation of the sphere may always be compounded from two successive rotations, one about the diameter normal to the plane, and the other about the diameter parallel to the ξ axis.

Now we learn from the group theory of three-dimensional rotations[3] that any unitary two-by-two matrix with unit determinant may be associated uniquely with a rotation of a sphere. The elements of this matrix are referred to as the Cayley-Klein parameters of the rotation. If we express the matrix of the optical system of Theorem I in the form **U**, then the Eulerian angles specifying the rotation of Poincaré's sphere are linearly related to the

[2] H. Poincaré, *Théorie Mathématique de la Lumière* (Paris, 1892), Vol. II, Chapter XII.

[3] Carl Eckart, Rev. Mod. Phys. **2**, 305 (1930), pp. 341, 346.

angles ϕ, θ and ψ. This is the relation between Poincaré's geometrical proof and our proof by the use of matrices.

From a purely formal point of view, the relation between the proofs consists in the fact that the group of unitary two-by-two matrices with unit determinant is isomorphic to the group of three-dimensional rotations.

Proof of Theorem II

We saw in Part I that all of the matrices representing partial polarizers and rotators could always be so written that the matrix elements were real and the determinants real and non-negative, except for phase factors occurring as factors of the matrix as a whole. For an optical system composed only of partial polarizers and rotators, we may therefore set

$$\mathbf{M}^{(n)} = e^{i\alpha}\mathbf{R}, \tag{15}$$

where $\mathbf{R}$ is a matrix whose matrix elements are real and whose determinant lies in the range zero to unity.

The matrix of a partial polarizer with the orientation ω is $\mathbf{S}(\omega)\mathbf{PS}(-\omega)$, where $\mathbf{P}$ is defined by Eq. (26, I). By the same argument that was used in the proof of Theorem I, the optical system composed of a rotator and a partial polarizer will always be represented by a matrix of the form

$$\mathbf{T} \equiv \mathbf{S}(A)\mathbf{PS}(B), \tag{16}$$

where A and B are related in an obvious manner to ω and $\tilde{\omega}$.

In order to establish the theorem, we must show that any matrix $\mathbf{R}$ which satisfies the stated conditions may be written in the form $\mathbf{T}$. We multiply out explicitly the expression on the right of (16), so that the matrix elements of $\mathbf{T}$ appear as explicit functions of A, B, p_1, and p_2. We then set the four elements of $\mathbf{R}$ equal, respectively, to the four elements of $\mathbf{T}$, and after a little algebraic reduction, we find

$$\begin{aligned}
(p_1+p_2)\sin(A+B) &= r_3-r_4,\\
(p_1+p_2)\cos(A+B) &= r_1+r_2,\\
(p_1-p_2)\sin(A-B) &= r_3+r_4,\\
(p_1-p_2)\cos(A-B) &= r_1-r_2
\end{aligned} \tag{17}$$

as the conditions which must be satisfied in order that $\mathbf{T}=\mathbf{R}$.

The quotient of the first two, and the quotient of the last two of the Eqs. (17) are

$$\tan(A+B) = (r_3-r_4)/(r_1+r_2), \tag{18a}$$

$$\tan(A-B) = (r_3+r_4)/(r_1-r_2). \tag{18b}$$

By the use of the trigonometric identity

$$\sin x \cos x \equiv \tan x/(1+\tan^2 x),$$

the product of the first two, and the product of the last two of the Eqs. (17) may be written

$$\begin{aligned}
p_1p_2 &= r_1r_2-r_3r_4 \equiv \det \mathbf{R},\\
p_1^2+p_2^2 &= r_1^2+r_2^2+r_3^2+r_4^2.
\end{aligned} \tag{19}$$

The Eqs. (18) always permit a solution for A and B, and, since the determinant of $\mathbf{R}$ is non-negative, the Eqs. (19) always permit a solution for p_1 and p_2 in which both p_1 and p_2 are positive. Although the Eqs. (19) always permit a solution for p_1 and p_2, they do not determine which is which; if $p_1=a$, $p_2=b$ is a solution of (19), then so is $p_1=b$, $p_2=a$. We must go back to the last two of the Eqs. (17) in order to determine which of the p's is the larger, and it is clear that the decision will depend on which of the two physically distinct solutions of (18b) has been chosen. Note also that only one of the two possible solutions of (18a) is admissible, since p_1+p_2 cannot be negative.

The theorem is thus established, provided only that we can show that neither of the p's is greater than unity. We proceed with this final part of the proof.

With every matrix $\mathbf{A}$ there may be associated a number $\Gamma(\mathbf{A})$ where Γ is the maximum value of the ratio $|\mathbf{A}\varepsilon|/|\varepsilon|$ with respect to any variation of the two components of ε. Such a maximum will always exist for a matrix $\mathbf{A}$ whose matrix elements are finite and do not depend on ε. For any two matrices $\mathbf{A}$ and $\mathbf{B}$, we obviously have the relation

$$\Gamma(\mathbf{AB}) \leqslant \Gamma(\mathbf{A})\Gamma(\mathbf{B}). \tag{20}$$

Since a unitary matrix does not change the length of a vector, we have

$$\begin{aligned}
\Gamma(\mathbf{U}) &= 1\\
\Gamma(\mathbf{AU}) &= \Gamma(\mathbf{UA}) = \Gamma(\mathbf{A}),
\end{aligned} \tag{21}$$

where $\mathbf{A}$ is any matrix and $\mathbf{U}$ is a unitary matrix.

The physical interpretation of Γ for a matrix which represents an element of an optical system is that Γ^2 is the maximum transmission factor of the given optical element. The value of Γ for such a matrix may therefore never be greater than unity. By repeated application of (20), we then find that the value of $\Gamma(\mathbf{M}^{(n)})$ may not be greater than unity. We clearly have $\Gamma(\mathbf{M}^{(n)})=\Gamma(\mathbf{R})$ and, since we have set $\mathbf{T}=\mathbf{R}$, we have also $\Gamma(\mathbf{M}^{(n)})=\Gamma(\mathbf{T})$. But by repeated application of (21), we have $\Gamma(\mathbf{T})=\Gamma(\mathbf{P})$, so that the value of Γ for $\mathbf{P}$ may not be greater than unity. The value of Γ for the matrix $\mathbf{P}$, however, is clearly the larger of the two p's, so that neither of the p's may be greater than unity. The proof of Theorem II is now complete.

Proof of Theorem III

In this section we shall show that any optical system composed of retardation plates, partial polarizers and rotators may always be replaced by an optical system composed of four elements; the four-element system consists of a partial polarizer placed between two retardation plates, with the addition of a rotator inserted at any of the four possible positions in the system.

We showed in Part I that, except for a phase factor, the matrix $\mathbf{M}^{(n)}$ representing the optical system could always be written so that the value of its determinant was real and in the range zero to unity. We therefore set

$$\mathbf{M}^{(n)}=e^{i\alpha}\mathbf{W}, \tag{22}$$

where $\mathbf{W}$ has a determinant which is real and which lies in the range zero to unity.

The matrix representing a partial polarizer placed between two retardation plates is

$$[\mathbf{S}(\omega_3)\mathbf{G}_1\mathbf{S}(-\omega_3)][\mathbf{S}(\omega_2)\mathbf{P}\mathbf{S}(-\omega_2)] \times[\mathbf{S}(\omega_1)\mathbf{G}_2\mathbf{S}(-\omega_1)]. \tag{23}$$

This matrix may be written in the form

$$\mathbf{\Phi}\equiv\mathbf{S}(A_1)\mathbf{G}_1\mathbf{S}(B_1)\mathbf{P}\mathbf{S}(A_2)\mathbf{G}_2\mathbf{S}(\mathbf{B}_2), \tag{24}$$

where the A's and B's are subject to the condition

$$A_1+B_1+A_2+B_2=0. \tag{25}$$

If now we add a rotator to the three-element system just described it is evident that it is always possible to choose the three ω's and the angle $\tilde{\omega}$ of the rotator so that the A's and B's may all be chosen arbitrarily. That is to say, the addition of a rotator permits us to avoid the restriction (25).

We thus see that the four-element optical system just described may be represented by the matrix $\mathbf{\Phi}$, where the A's and B's are now all independent. The matrix $\mathbf{\Phi}$ is of the form $\mathbf{V}_1\mathbf{P}\mathbf{V}_2$, where $\mathbf{V}$ is the matrix defined by Eq. (4). We have shown that any matrix of the form $\mathbf{V}$ is mathematically equivalent to a matrix of the form $\mathbf{U}$, where $\mathbf{U}$ is defined by Eq. (2). Any matrix of the form $\mathbf{\Phi}$ may therefore be written in the form $\mathbf{U}_1\mathbf{P}\mathbf{U}_2$, *and conversely.*

In order to establish the theorem, we must now show that any matrix $\mathbf{W}$ can be written in the form

$$\mathbf{\Phi}\equiv\mathbf{U}_1\mathbf{P}\mathbf{U}_2. \tag{26}$$

This fact will be proved in the following paragraphs. In anticipation of this result, we should like to make an important observation: The matrices $\mathbf{W}$ and $\mathbf{\Phi}$ are not completely general, because they are restricted by the fact that the imaginary parts of their determinants are zero. They therefore contain eight minus one, or seven *independent* variables. The matrix $\mathbf{\Phi}$, however, contains eight variables (three in each of the $\mathbf{U}$'s and two in $\mathbf{P}$), so that these variables cannot be uniquely determined. We shall find that the extra degree of freedom afforded by the eight variables will often, but by no means always, permit us to dispense with the rotator; whenever the extra degree of freedom permits us to satisfy the relation (25), the rotator is unnecessary.

We now multiply out explicitly the product on the right in Eq. (26). We find

$$\mathbf{\Phi}=\begin{vmatrix} \exp[i(\phi_1+\phi_2)]p_1\cos\theta_1\cos\theta_2 & -\exp[i(\phi_1-\psi_2)]p_1\cos\theta_1\sin\theta_2 \\ -\exp[-i(\psi_1-\psi_2)]p_2\sin\theta_1\sin\theta_2 & -\exp[-i(\phi_2+\psi_1)]p_2\sin\theta_1\cos\theta_2 \\ \exp[i(\phi_2+\psi_1)]p_1\sin\theta_1\cos\theta_2 & -\exp[i(\psi_1-\psi_2)]p_1\sin\theta_1\sin\theta_2 \\ +\exp[-i(\phi_1-\psi_2)]p_2\cos\theta_1\sin\theta_2 & +\exp[-i(\phi_1+\phi_2)]p_2\cos\theta_1\sin\theta_2 \end{vmatrix}. \tag{27}$$

By examination of this matrix, we see that the ϕ's and ψ's occur only in four different combinations

$$\begin{aligned} &\chi_1 \equiv \phi_1 + \phi_2, \quad \chi_3 \equiv \phi_2 + \psi_1, \\ &\chi_2 \equiv \psi_1 - \psi_2, \quad \chi_4 \equiv \phi_1 - \psi_2. \end{aligned} \tag{28}$$

These four relations are not linearly independent, in view of the relation

$$\chi_1 + \chi_2 = \chi_3 + \chi_4.$$

We therefore choose to replace the ϕ's and ψ's by three new *independent* variables η, ξ_1, and ξ_2:

$$\begin{aligned} &\phi_1 + \phi_2 = \eta + \xi_1, \quad \phi_2 + \psi_1 = \eta + \xi_2, \\ &\psi_1 - \psi_2 = \eta - \xi_1, \quad \phi_1 - \psi_2 = \eta - \xi_2. \end{aligned} \tag{29}$$

We have the following definitions of η, ξ_1, and ξ_2:

$$\begin{aligned} 2\xi_1 &\equiv \phi_1 + \phi_2 - \psi_1 + \psi_2, \\ 2\xi_2 &\equiv -\phi_1 + \phi_2 + \psi_1 + \psi_2, \\ 2\eta &\equiv \phi_1 + \phi_2 + \psi_1 - \psi_2, \\ 2\sigma &\equiv \phi_1 - \phi_2 + \psi_1 + \psi_2, \end{aligned} \tag{30}$$

where we have added a fourth variable σ in order that we may also solve the equations for the ϕ's and ψ's:

$$\begin{aligned} &2\phi_1 = \xi_1 - \xi_2 + \eta + \sigma, \quad 2\psi_1 = -\xi_1 + \xi_2 + \eta + \sigma, \\ &2\phi_2 = \xi_1 + \xi_2 + \eta - \sigma, \quad 2\psi_2 = \xi_1 + \xi_2 - \eta + \sigma. \end{aligned} \tag{31}$$

We make the further changes of variable

$$q_1 \equiv e^{i\eta} p_1, \quad q_2 \equiv e^{-i\eta} p_2. \tag{32}$$

The q's are thus complex quantities.

With the changes of variable outlined in the last paragraph, the matrix Φ may now be written in the simpler form

$$\Phi = \begin{pmatrix} \exp(i\xi_1)(q_1 \cos\theta_1 \cos\theta_2 - q_2 \sin\theta_1 \sin\theta_2) & -\exp(-i\xi_2)(q_1 \cos\theta_1 \sin\theta_2 + q_2 \sin\theta_1 \cos\theta_2) \\ \exp(i\xi_2)(q_1 \sin\theta_1 \cos\theta_2 + q_2 \cos\theta_1 \sin\theta_2) & -\exp(-i\xi_1)(q_1 \sin\theta_1 \sin\theta_2 - q_2 \cos\theta_1 \cos\theta_2) \end{pmatrix}. \tag{33}$$

We now set the four matrix elements of Φ equal to the four matrix elements of $\mathbf{W}$. The resulting four equations are equivalent to

$$\begin{aligned} (q_1+q_2)\sin(\theta_1+\theta_2) &= \exp(-i\xi_2)w_3 - \exp(i\xi_2)w_4, \\ (q_1+q_2)\cos(\theta_1+\theta_2) &= \exp(-i\xi_1)w_1 + \exp(i\xi_1)w_2, \\ (q_1-q_2)\sin(\theta_1-\theta_2) &= \exp(-i\xi_2)w_3 + \exp(i\xi_2)w_4, \\ (q_1-q_2)\cos(\theta_1-\theta_2) &= \exp(-i\xi_1)w_1 - \exp(i\xi_1)w_2, \end{aligned} \tag{34}$$

where the w's are the matrix elements of $\mathbf{W}$, and are in general complex.

We wish to show that these equations can always be solved by θ's and ξ's which are real, and by q's whose product is real and non-negative. The quotient of the first two, and the quotient of the last two of the above equations are

$$\begin{aligned} \tan(\theta_1+\theta_2) &= \frac{\exp(-i\xi_2)w_3 - \exp(i\xi_2)w_4}{\exp(-i\xi_1)w_1 + \exp(i\xi_1)w_2}, \\ \tan(\theta_1-\theta_2) &= \frac{\exp(-i\xi_2)w_3 + \exp(i\xi_2)w_4}{\exp(-i\xi_1)w_1 - \exp(i\xi_1)w_2}. \end{aligned} \tag{35}$$

In order that these equations be capable of solution by real values of the θ's, it is necessary that the imaginary parts of the expressions on the right side be zero. By rationalizing the denominators, we find that this condition is equivalent to

$$\text{I.P. } \{w_1^* w_3 \exp[i(\xi_1 - \xi_2)] - w_2^* w_4 \exp[-i(\xi_1 - \xi_2)]\} = 0,$$

$$\text{I.P. } \{w_1^* w_4 \exp[i(\xi_1 + \xi_2)] - w_2^* w_3 \exp[-i(\xi_1 + \xi_2)]\} = 0$$

or finally,

$$\begin{aligned}\xi_1-\xi_2&=\arg\,(w_1w_3^*+w_2^*w_4),\\ \xi_1+\xi_2&=\arg\,(w_1w_4^*+w_2^*w_3).\end{aligned}\tag{36}$$

The last two equations serve to evaluate the ξ's, and with this choice of the ξ's, Eqs. (35) may always be solved for real values of the θ's.

We may now determine the q's from the Eqs. (34). In order, however, to show that the product q_1q_2 is real and non-negative, we form the product of the first two and the product of the last two of the Eqs. (34), and then subtract the two resulting equations. With the help of (35), we find the result reduces to

$$q_1q_2=w_1w_2-w_3w_4=\det\mathbf{W}.\tag{37}$$

The matrix $\mathbf{W}$ was obtained from $\mathbf{M}^{(n)}$ by requiring that the determinant of $\mathbf{W}$ be real and non-negative. After evaluating q_1 and q_2, we may determine the quantity η from the condition that p_1 and p_2 be individually real and non-negative. (Cf. Eq. (32).)

Up to this point we have shown that the equations obtained by setting $\mathbf{\Phi}$ equal to $\mathbf{W}$ can always be solved for the real quantities ξ_1, ξ_2, θ_1, θ_2, p_1, p_2 and η, in the order mentioned. If we assign an arbitrary real value to σ, the Eqs. (31) may always be solved for the ϕ's and ψ's. The proof that neither of the p's is greater than unity is exactly similar to the proof used in Theorem II. The present theorem is therefore established.

It remains to determine the condition that we be able to choose σ so that the equation

$$A_1+B_1+A_2+B_2=0\tag{25}$$

is satisfied. If σ can be so chosen, the rotator is unnecessary. Now let us consider the equation

$$\tan\,(A_1+B_1)+\tan\,(A_2+B_2)=\text{constant}.\tag{38}$$

In order that we be able to choose the A's and B's so that they satisfy (25), it is necessary and sufficient that the constant in Eq. (38) be zero. The validity of the equation

$$\tan\,(A_1+B_1)+\tan\,(A_2+B_2)=0\tag{39}$$

is therefore the necessary and sufficient condition that we be able to dispense with the rotator.

By the use of Eqs. (31) and Eq. (6), we find that (39) may be written:

$$\cos\psi_1\cos\phi_2\tan\theta_1+\cos\psi_2\cos\phi_1\tan\theta_2=0\tag{40}$$

or

$$\cos\tfrac{1}{2}[-\xi_1+\xi_2+\eta+\sigma]\cos\tfrac{1}{2}[\xi_1+\xi_2+\eta-\sigma]\times\tan\theta_1+\cos\tfrac{1}{2}[\xi_1+\xi_2-\eta+\sigma]\cos\tfrac{1}{2}[\xi_1-\xi_2+\eta+\sigma]\tan\theta_2=0.\tag{41}$$

By trigonometric reduction we find that the last equation is equivalent to

$$k_1\cos\sigma+k_2\sin\sigma+k_3=0,\tag{42}$$

where

$$\begin{aligned}k_1&\equiv\cos\xi_1(\tan\theta_1+\tan\theta_2)\\ k_2&\equiv\sin\xi_1(\tan\theta_1-\tan\theta_2)\\ k_3&\equiv\cos\,(\eta+\xi_2)\tan\theta_1+\cos\,(\eta-\xi_2)\tan\theta_2.\end{aligned}\tag{43}$$

The condition which must be satisfied in order that Eq. (42) be solvable for σ is

$$k_3^2\leqslant k_1^2+k_2^2.\tag{44}$$

This inequality is the condition which must be satisfied in order that it be possible to dispense with the rotator. As it is written, the condition is purely a mathematical one. Although the condition can be simplified slightly, the authors have not been able to attach any simple physical significance to the inequality expressed by Eq. (44).

A New Calculus for the Treatment of Optical Systems

III. The Sohncke Theory of Optical Activity

R. CLARK JONES*
Research Laboratory, Polaroid Corporation, Cambridge, Massachusetts, and Research Laboratory of Physics, Harvard University, Cambridge, Massachusetts

(Received April 23, 1941)

Reusch and Sohncke have examined the properties of a system containing a large number n of identical retardation plates, each of which is rotated with respect to the one preceding it through the angle ω. The product of n and ω must be equal to $\mu\pi$, where μ is an integer. Under certain conditions this system is optically equivalent to a simple rotator. This system, which would be very difficult to examine by ordinary methods, is given a treatment which is completely rigorous and which is more general than any given heretofore.

INTRODUCTION

REUSCH[1] has examined experimentally the optical properties of a system composed of n identical retardation plates, the corresponding axes of which are distributed uniformly about a whole number of semicircles. If the number of semicircles is denoted by μ, we may describe the system by stating that the axis of each plate is rotated through the angle $-\pi\mu/n$ with respect to the corresponding axis of the plate immediately preceding it. Reusch found experimentally that when the phase retardation of each of the plates was small, the complete optical system behaved approximately like a simple rotator. A few years after Reusch published his results, Sohncke[2] provided a rigorous mathematical treatment of the system composed of three retardation plates with their axes distributed through 180°—that is to say, with the relative angle 60°. In his book[3] on crystal optics, Pockels derives an approximate expression for the angle of the rotation produced by the system of n retardation plates distributed through 180°:

$$\bar{\omega}=\tfrac{1}{8}n\Delta^2\cot(\pi/n). \qquad (1)$$

In this expression, Δ is the phase retardation of each of the plates, and is assumed to be small. Just how small Δ must be, will be indicated in the treatment to be given.

We present here a rigorous treatment of the optical system described at the beginning of the previous paragraph. The results will be exact for arbitrary integral values of n and μ. By use of the transformation properties of matrices, we shall find an exact expression for the matrix $\mathbf{M}^{(n)}$ of the entire optical system. Since the optical system under consideration contains only retardation plates, it satisfies the conditions of Theorem I of Part II, and may therefore be replaced by a system composed only of one rotator and one retardation plate. We shall find exact and reasonably simple expressions (cf. Eqs. (18) and (20)) for the angle $\bar{\omega}$ of the equivalent rotator and the phase retardation $2\bar{\gamma}$ of the equivalent retardation plate, and we shall derive the condition (cf. Eq. (21)) which must be satisfied in order that the optical system behave approximately as a rotator. Our results will confirm Pockels' approximate result, Eq. (1).

DETERMINATION OF THE MATRIX $\mathbf{M}^{(n)}$

With respect to the fixed axes x and y, let the orientation of the plate which the light enters first be zero. Then in general the orientation of the kth plate will be

$$\omega_k=-(k-1)\pi\mu/n. \qquad (2)$$

Let us represent the retardation plate in the form $\mathbf{G}$ (cf. Eq. (25, I)). The phase retardation of the plate is then $\Delta=2\gamma$. According to Eqs. (19, I) and

* National Scholar, Harvard University.

[1] E. Reusch, Berliner Monatsber. (Gesammtsitzung, July 8, 1869) 530 (1869); Pogg. Ann. **138**, 628 (1869).

[2] L. Sohncke, Math. Ann. **9**, 504 (1876). See also Pogg. Ann. Ergänzungsband **8**, 16 (1878), and Zeits. f. Krist. **13**, 214 (1888).

[3] F. Pockels, *Lehrbuch der Kristalloptik* (Teubner, Leipzig, 1906), pp. 289–290.

(20, I), the matrix $\mathbf{M}^{(n)}$ of the complete optical system reduces to

$$\begin{aligned}\mathbf{M}^{(n)} &= \mathbf{S}(\pi\mu)(\mathbf{S}(\omega)\mathbf{G})^n \\ &= (-1)^\mu(\mathbf{S}(\omega)\mathbf{G})^n,\end{aligned} \tag{3}$$

where ω is here minus the relative orientation of the plates

$$\omega \equiv \pi\mu/n. \tag{4}$$

The determination of $\mathbf{M}^{(n)}$ thus reduces to the problem of finding the nth power of a matrix which in this case is unitary.

Now there exists a simple method of finding the nth power of any matrix which can be diagonalized. Let $\mathbf{A}$ be any matrix which can be diagonalized, and let $\mathbf{T}$ be the matrix of the transformation which accomplishes the diagonalization:

$$\begin{aligned}\mathbf{D} &= \mathbf{TAT}^{-1}, \\ \mathbf{A} &= \mathbf{T}^{-1}\mathbf{DT},\end{aligned} \tag{5}$$

where $\mathbf{D}$ is the diagonal form of $\mathbf{A}$, and where $\mathbf{T}^{-1}$ is the reciprocal of $\mathbf{T}$:

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{TT}^{-1} = 1. \tag{6}$$

The nth power of $\mathbf{A}$ is then

$$\begin{aligned}\mathbf{A}^n &= \mathbf{T}^{-1}\mathbf{DTT}^{-1}\mathbf{DT}\cdots\mathbf{T}^{-1}\mathbf{DT} \\ &= \mathbf{T}^{-1}\mathbf{D}^n\mathbf{T}.\end{aligned} \tag{7}$$

Since the nth power of a diagonal matrix can be written down at once, the determination of $\mathbf{A}^n$ reduces to the problem of determining the transformation $\mathbf{T}$.

The writer has found by detailed calculation the following result. Let a matrix $\mathbf{A}$ be diagonalized by the transformation $\mathbf{T}$, and let d_1 and d_2 be the diagonal elements of the matrix $\mathbf{D}$ so obtained. Let now d_1 and d_2 be replaced by new diagonal elements d_1' and d_2', and let the resulting matrix be transformed back by the reciprocal of the transformation $\mathbf{T}$. We denote the matrix so obtained by $\mathbf{A}'$. The matrix elements of $\mathbf{A}'$ are then related to those of $\mathbf{A}$ by the relations

$$\begin{aligned}a_1' &= \frac{d_1'-d_2'}{d_1-d_2}a_1 + \frac{d_2'd_1-d_1'd_2}{d_1-d_2}, \\ a_2' &= \frac{d_1'-d_2'}{d_1-d_2}a_2 + \frac{d_2'd_1-d_1'd_2}{d_1-d_2}, \\ a_3' &= \frac{d_1'-d_2'}{d_1-d_2}a_3. \\ a_4' &= \frac{d_1'-d_2'}{d_1-d_2}a_4.\end{aligned} \tag{8}$$

These relations involve only the old and new diagonal elements, and do not involve the transformation $\mathbf{T}$.

In application to Eq. (3), the matrix $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{S}(\omega)\mathbf{G} = \begin{pmatrix} e^{i\gamma}\cos\omega & -e^{-i\gamma}\sin\omega \\ e^{i\gamma}\sin\omega & e^{-i\gamma}\cos\omega \end{pmatrix}. \tag{9}$$

The diagonal elements, or characteristic values, of this matrix are the roots of

$$(e^{i\gamma}\cos\omega - d)(e^{-i\gamma}\cos\omega - d) + \sin^2\omega = 0 \tag{10}$$

or

$$d_1 = e^{i\chi}, \quad d_2 = e^{-i\chi}, \tag{11}$$

where

$$\cos\chi = \cos\omega\cos\gamma. \tag{12}$$

For the sake of definiteness, we assume that χ is the positive solution of this equation. The matrix $\mathbf{D}^n$ has the diagonal elements $e^{in\chi}$ and $e^{-in\chi}$, so that if we set

$$d_1' = e^{in\chi}, \quad d_2' = e^{-in\chi}, \tag{13}$$

then the matrix $\mathbf{A}'$ is equal to $\mathbf{A}^n$.

By the indicated procedure, we find

$$\mathbf{M}^{(n)} = (-1)^\mu\mathbf{A}^n = (-1)^\mu\mathbf{A}', \tag{14}$$

where

$$\begin{aligned}a_1' &= \frac{\sin n\chi}{\sin\chi}\cos\omega e^{i\gamma} - \frac{\sin(n-1)\chi}{\sin\chi}, \\ a_2' &= \alpha_1'^*, \\ a_3' &= \frac{\sin n\chi}{\sin\chi}\sin\omega e^{i\gamma}, \\ a_4' &= -a_3'^*.\end{aligned} \tag{15}$$

We find that these expressions for the matrix elements of $\mathbf{A}'$ yield the following expressions for the elements of $\mathbf{M}^{(n)}$, which we denote simply by m_1, m_2, m_3 and m_4:

$$\begin{aligned} m_1 &= \cos n(\chi-\omega)+i[\sin n(\chi-\omega)]\frac{\cos\omega\sin\gamma}{\sin\chi}, \\ m_2 &= m_1^*, \\ m_3 &= [\sin n(\chi-\omega)]\frac{\sin\omega}{\sin\chi}e^{i\gamma}, \\ m_4 &= -m_3^*. \end{aligned} \tag{16}$$

We should like to stress that these expressions are exact, no approximations having been used in their derivation.

Resolution of the Matrix $\mathbf{M}^{(n)}$

It is evident that the matrix $\mathbf{M}^{(n)}$ does not represent a pure rotation, since the matrix elements of a rotator are real. We wish to know, however, under just what conditions and in just what sense we may say that the matrix $\mathbf{M}^{(n)}$ is an approximation to a rotator. The most direct and elegant way to determine these conditions is to express the matrix $\mathbf{M}^{(n)}$ as the combination of a rotator and a retardation plate. We know by Theorem I of Part II that this may always be done.

We shall express the matrix $\mathbf{M}^{(n)}$ as the combination of a rotator $\mathbf{S}(\tilde{\omega})$ and a retardation plate $\bar{\mathbf{G}}$ with the orientation $\tilde{\omega}$:

$$\mathbf{M}^{(n)}=\mathbf{S}(\tilde{\omega})[\mathbf{S}(\tilde{\omega})\bar{\mathbf{G}}\mathbf{S}(-\tilde{\omega})]. \tag{17}$$

From Eqs. (6–8, II), (12) and (16), we find

$$\begin{aligned} \tan\tilde{\omega}=\tan(A+B) &= \frac{m_3-m_4}{m_1+m_2} \\ &= [\tan n(\chi-\omega)]\frac{\tan\omega}{\tan\chi}, \end{aligned} \tag{18}$$

$$\begin{aligned} \tan(\tilde{\omega}+2\tilde{\omega})=\tan(A-B) &= \frac{m_3+m_4}{m_1-m_2} \\ &= \tan\omega, \end{aligned} \tag{19}$$

$$\begin{aligned} \tan\bar{\gamma} &= \frac{\sin(A+B)}{\sin(A-B)}\tan\gamma \\ &= \sin\tilde{\omega}\frac{\tan\gamma}{\sin\omega}. \end{aligned} \tag{20}$$

We have found that the complete optical system is equivalent to the combination of a rotator with angle $\tilde{\omega}$ and a retardation plate with the relative phase retardation $2\bar{\gamma}$. In order that the matrix $\mathbf{M}^{(n)}$ represent a rotator, it is therefore reasonable to require that $\bar{\gamma}$ must be small compared with $\tilde{\omega}$, and in order to insure this inequality, it is by Eq. (20) necessary and sufficient to require that

$$\left|\frac{\tan\gamma}{\sin\omega}\right|\ll 1. \tag{21}$$

This condition may be more stringent than the condition that γ be small compared with unity, because Eq. (21) is equivalent to the condition

$$\gamma\ll\omega=\pi\mu/n. \tag{22}$$

The following two relations may be derived from the Eq. (12) defining χ:

$$\begin{aligned} \sin(\chi-\omega) &= \sin\omega\cos\omega\cos\gamma \\ &\quad\times\left[\left(1+\frac{\tan^2\gamma}{\sin^2\omega}\right)^{\frac{1}{2}}-1\right], \end{aligned} \tag{23}$$

$$\frac{\tan^2\chi}{\tan^2\omega}=1+\frac{\tan^2\gamma}{\sin^2\omega}. \tag{24}$$

When the condition (21) is satisfied, by Eqs. (18), (23) and (24) we then have for the angle $\tilde{\omega}$:

$$\begin{aligned} \tilde{\omega} &\cong n(\chi-\omega)\cong n\sin(\chi-\omega) \\ &\cong \tfrac{1}{2}n\cot\omega\sin\gamma\tan\gamma\cong\tfrac{1}{2}n\gamma^2\cot\omega \end{aligned} \tag{25}$$

or

$$\tilde{\omega}=O(n^2\gamma^2/2\pi\mu). \tag{26}$$

The approximately equal sign in these equations should be understood to mean that the ratio of the expressions on the two sides approaches unity as $\tan\gamma/\sin\omega$ approaches zero.

Comparing (22) with (26), we find that the rotation $\tilde{\omega}$ is subject to the condition

$$\tilde{\omega}\ll\tfrac{1}{2}\pi\mu. \tag{27}$$

We see from this relation that in order to obtain a rotation which is not infinitesimal, it is necessary that $\mu\gg 1$—that is to say, that the axes of the plates be distributed uniformly through a large number of semicircles.

It should be noted that whereas the orientations of the plates advance in the clockwise

direction, the angle of the equivalent rotator is in the counter clockwise direction if $\cot \omega$ is positive.

Rotator Formed by Three Retardation Plates

The considerations presented so far in this part had at one time considerable theoretical interest in connection with the theory of crystal structure.[4] If one wishes to form a rotator from retardation plates alone, however, it may be done in a much simpler manner than might be inferred from the foregoing. In fact, it is possible to produce an optical system which behaves rigorously as a rotator with any given $\tilde{\omega}$, by the use of only three retardation plates.

We may produce a rotator by inserting a retardation plate between two crossed quarter-wave plates in such a position that its axes are at 45° with respect to the axes of the quarter-wave plates. Let us write the matrix defined by Eq. (25, I) as $\mathbf{G}(\gamma)$. The statement we have just made then follows directly from the identity

$$\mathbf{S}(ab\gamma) \equiv \mathbf{G}(a\pi/4)[\mathbf{S}(b\pi/4)\mathbf{G}(\gamma) \times \mathbf{S}(-b\pi/4)]\mathbf{G}(-a\pi/4), \quad (28)$$

where

$$a = \pm 1, \quad b = \pm 1.$$

Since the relative phase retardation of the middle plate is 2γ, we see that the angle of the rotation is one-half of the phase retardation of the inserted plate.

By the use of two achromatic quarter-wave plates, it is possible in this manner to use a Soleil compensator as a variable rotator.[5]

There exists another theorem which may be considered as the converse of the one just proved. We may produce a retardation plate of any given retardation from two quarter-wave plates and a rotator. The rotator should be placed between the two crossed quarter-wave plates. The optical system then behaves as a retardation plate with its axes at 45° with those of the quarter-wave plates. This statement follows directly from the identity

$$\mathbf{G}(ab\tilde{\omega}) \equiv \mathbf{S}(a\pi/4)[\mathbf{G}(b\pi/4)\mathbf{S}(\tilde{\omega}) \times \mathbf{G}(-b\pi/4)]\mathbf{S}(-a\pi/4), \quad (29)$$

where

$$a = \pm 1, \quad b = \pm 1.$$

We see that the phase retardation of the optical system is twice the angle of the rotator.

With two achromatic quarter-wave plates, it is thus possible to use a variable rotator as a Soleil compensator.

According to Billet,[6] the physical statements corresponding to the identities (28) and (29) were known to Fresnel.

Conclusion

Part IV of this series, which is now in preparation, will contain, in addition to emendations to the general theory, derivations of the matrices which represent (1) plates or solutions exhibiting circular birefringence or circular dichroism, and (2) plates cut from crystals of such low symmetry (monoclinic and triclinic) that the principal axes of absorption are not parallel to the principal axes of refraction.

The writer is indebted to Mr. Edwin H. Land, President and Research Director of the Polaroid Corporation, for suggesting the problem of developing an operational calculus of the type presented here. He wishes to thank Dr. Cutler D. West, also of the Polaroid Corporation, for providing all of the references to the older optical literature, and he is indebted to his co-author of Part II, Dr. Henry Hurwitz, Jr., for helpful suggestions about Parts I and III. The author wishes to express his appreciation to Professor M. S. Vallarta, Department of Physics, Massachusetts Institute of Technology, for reading the first draft of the manuscripts and for making helpful suggestions.

[4] L. Sohncke, *Entwickelung einer Theorie der Krystallstructur* (Teubner, Leipzig, 1879). See particularly pp. 238–246.

[5] T. M. Lowry, *Optical Rotatory Power* (Longmans Green, New York, 1935), p. 194, Fig. 79(a).

[6] M. F. Billet, *Traité D'optique Physique* (Paris, 1859), Vol. 2, Chapter 20, Art. 1, pp. 328–351.

Photoelectric Cells for the Visible Spectral Range

P. GÖRLICH
Dresden, Germany
(Received April 30, 1941)

WHILE nearly all technical problems dealing with the short wave infra-red (from 6000 to 9000A) can be solved by using a Cs_2O photocathode, there has existed until recently no photoelectric cell with a suitable high sensitivity for the visible range of wave-lengths shorter than 6000A. The previously used hydrated potassium cells could not be considered as a reliable means for this purpose on account of their low sensitivity and instability. The author has succeeded in solving this problem by using a new cathode layer,[1] the properties of which have been described elsewhere.

We are here concerned with cathodes having this new structure, which contain electron-emitting layers consisting of alloys of an alkali metal and a metal of lower conductivity; for example, bismuth or antimony. The layer can be applied in the form of transparent films on the surface of the bulb, or in the form of thick opaque films on a metallic base, usually silver.[2] The sensitivity of films equipped with such layers is surprisingly high. Even in the routine manufacturing process, 60 microamperes per lumen are obtained for light of a color temperature of 2360°K. Recently Glover and Janes[3] have referred to cells with similar high sensitivity in the visible spectral range. The purpose of the present note is to point out that the cathodes mentioned by Glover and Janes are also alloy cathodes of a type previously described by the author.

Alloy cathodes show a selective maximum between 4000 and 5000A, whereas Cs_2O cathodes have a maximum between 7500 and 8500A. When the Cs_2O cathode is overlaid by the alloy layer,[4] and a suitable thickness of the alloy layer is chosen, there results an over-all sensitivity which shows the maximum of the alloy cathode, as well as the maximum of the Cs_2O layer, but extending somewhat in the direction of short wave-lengths. By this means it is possible to produce light sensitive cells with a nearly constant sensitivity over the entire visible spectral range. Obviously electrons from the deeper lying layers of Cs_2O are selectively transmitted through the alloy layers. Measurements indicate that the transmission of electrons depends on the lattice structure of the alloy cathode, and that by the extension of the lattice of the alloy cathode a displacement of the Cs_2O maximum takes place towards longer wave-lengths.

Whereas in a Cs_2O cathode the metallic base has an influence on both the sensitivity and the spectral distribution, the base in an alloy cathode exerts an effect only when its transverse conductivity is not sufficient.[5] Alloy cathodes also behave differently from Cs_2O cathodes when contaminated by foreign metals. No method has been found of increasing the sensitivity of an alloy cathode by introduction of a foreign metal.

Most of the above-mentioned experiments were carried out with Sb—Cs alloys. Final tests showed that a similar effect is achieved when using Sb—Li alloys.[6] The Sb—Li layers show a

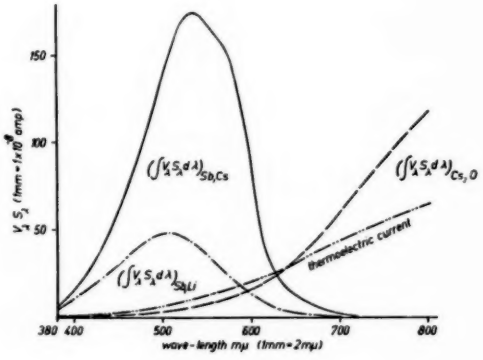

FIG. 1.

[1] U. S. Patent 2,122,860 (August, 1936); British Patent 460,012 (January, 1937).
[2] P. Görlich, Zeits. f. Physik **101**, 335 (1936); Zeits. f. tech. Physik **18**, 460 (1937).
[3] A. M. Glover and R. B. Janes, Electronics **13** (No. 8), 26 (1940).
[4] P. Görlich, Zeits. f. Physik **109**, 374 (1938); U. S. Patent 2,172,164 (June, 1937); British Patent 479,733 (February, 1938).
[5] P. Görlich, Zeits. f. Physik **116**, 704 (1940).
[6] P. Görlich, Phil. Mag. [7] **25**, 256 (1938); P. Görlich and E. J. Meyer, Zeits. f. Astrophys. **16**, 343 (1938).

TABLE I.

$\lambda_1 \cdots \lambda \cdots \lambda_2$ IN A	$\int V_\lambda S_\lambda d\lambda$			$\int V_\lambda d\lambda$ (S_λ = CONSTANT)		
	[Ag] −Sb, Cs −Cs	[Ag] −Sb, Li −Li	[Ag] $-Cs_2O$, Ag, Cs −Cs	[Ag] −Sb, Cs −Cs	[Ag] −Sb, Li −Li	[Ag] $-Cs_2O$, Ag, Cs −Cs
4000–8000	118.0	34.8	66.0	95.4	39.2	9.5
4000–7000	117.7	34.7	22.1	95.4	39.2	5.5
4150–7500	115.6	32.5	40.4	82.6	33.8	7.2

TABLE II.

ALLOY COMPONENT M	CONDUCTIVITY $\times 10^{-4}$ CM$^{-1}\Omega^{-1}$	QUANTA GAINED IN % (CATHODE: [Ag] −M, Cs −Cs)
Bi	0.88	∼10
Hg	1.04	∼ 0.5
Sb	2.56	∼30
Pb	4.91	∼ 2
Tl	5.28	∼ 1

TABLE III.

ELEMENT	ATOM WEIGHT	DENSITY IN G/CM^3	ATOM VOLUME
Cs	132.91	1.87	71.1
Rb	85.44	1.52	56.1
K	39.096	0.86	45.5
Na	22.997	0.97	23.6
Bi	209.00	9.80	21.3
Sb	121.76	6.69	18.2
Si	28.06	2.33	12.1
As	74.91	5.72	13.1
B	10.82	1.73	6.3
P	31.02	1.83	16.9

TABLE IV.

PROPORTION OF THE ATOMIC VOLUMES		QUANTA GAINED IN %
A_{Cs}/A_{Sb}	3.92	∼30
A_{Cs}/A_{Bi}	3.35	∼10
A_{Rb}/A_{As}	4.29	∼ 0.5
A_{Na}/A_{B}	3.76	∼ 0.5
A_{K}/A_{Si}	3.77	∼ 1
A_{Cs}/A_{P}	4.22	Investigated by another author.

relatively high sensitivity and a very low dark current.

A summary will now be given of the sensitivity of different layers with respect to each other. It is known that

$$J_{ph} = c \int V_\lambda S_\lambda d\lambda,$$

V_λ being the spectral distribution of the photocell and S_λ the distribution of energy in the light source at a known color temperature T_{col}. Integrals taken from Fig. 1 are shown in Table I ($T_{col} = 2180°K$).

It is evident that the alloy cathodes in the visible spectral range are superior to the Cs_2O cathodes and therefore are to be preferred.

Numerous investigations on the new cathodes show that the emission mechanism acting in the alloy cathodes is different from the one acting in the Cs_2O cathodes. First, it became apparent that in the case of alloy cathodes the conductivity of the component alloyed with the alkali metal was of great importance. According to the data of Table II, Bi, being a component of metallic qualities and of low conductivity, should produce the largest gain in quanta. Experiments have shown however that Sb produces better results.

A further method of investigating the mechanism of emission consists in investigating alloys in which the atomic weight of the alkali component does not differ greatly from the atomic weight of the other metal. In Table III the atomic weights, densities, and atomic volumes are given for the elements in question.

Besides the combinations Cs−Sb and Cs−Bi, the following combinations were investigated: (a) Rb−As, (b) Na−B, (c) K−Si.

In all the above-mentioned combinations the ratio of the atomic volume of the alkali metal component and the atomic volume of the other components is nearly the same (see Table IV).

It has been demonstrated by experiment that an extremely high gain of quanta can be obtained only by the combinations Cs−Sb and Cs−Bi. It was not possible from these results to determine more definitely the details of the emission mechanism. Perhaps it will be possible to obtain a complete answer to this problem when the lattice structure of the alloys employed becomes known.

LETTERS TO THE EDITOR

Note on Construction of the Photoelastic Interferometer

F. W. BUBB
Department of Applied Mathematics, Washington University, St. Louis, Missouri
May 20, 1941

I HAVE recently learned, through private correspondence, that several photoelastic laboratories propose constructing photoelastic interferometers along the lines described in my article[1] "A complete photoelastic instrument." I should like to comment on several points which have been raised concerning the construction and operation of this instrument.

In the first place, Dr. David Sinclair[2] has pointed out that the light must be rotated 90° uniformly over the field to within ±1°. My own instrument is not yet completed, but in case undesirable effects arise due to lack of uniformity, I had proposed to correct as follows. The magneto-optic law is $\theta = VHL$, where θ is the angle of rotation of the light, V is Verdet's constant, H is the field intensity, and L is the length of path in the rotating liquid core of the solenoid. If H varies (and only a radial variation is significant), the rotator may be corrected by varying L. This can be done by inserting a transparent compensating piece in the liquid core. The thickness of this compensator may be varied with the radius so that the length of light path in the rotating liquid varies with the radius—and in such a way as to give uniform rotation. The index of refraction of this compensator must, of course, be equal to that of the rotating liquid; otherwise, it would act as a lens and destroy the parallelism of the beam.

Comments have been made as to difficulties which might arise due to the size of the photoelastic interferometer. I think these are answered by the successful operation of the large interferometer used in the Michelson-Gale[3] "Clearing Experiment." I saw this experiment, and one arm of the interferometer was 2000 feet, the other 1000 feet.

As pointed out in my original article, portions of the face of the model become inclined in a region of high isopachic gradient. Such portion of the model then acts like a prism and refracts the light away from its original path. This deflection may be so much that the light, upon its return from the rotator, may pass through a different spot —thus blurring the isopachic. This difficulty may be avoided by immersing the model in a liquid of the same index of refraction, as pointed out in my original article. Dr. Sinclair[2] agrees with this, but points out that immersion is impractical for centrifugally loaded models. Problems involving centrifugal loading, although perhaps few in number, are important. It seems to me, however, that such problems may be solved by the stress freezing technique of Hetényi[4] and Newton.[5] Having frozen in the stress, the isopachic lines may be photographed with the photoelastic interferometer just as these lines may be photographed for static tests. Certainly the isochromatic and isoclinic lines have been photographed in this manner.

I have had a number of inquiries as to the necessary accuracy of the model. If the model be immersed in a liquid of the same index of refraction, the model does not have to be optically flat or optically parallel—in fact, it does not even have to be polished. If the model be not immersed, its order of accuracy should be about the same as that maintained for polariscope models. With the photoelastic interferometer, the light passes through the model, just as it does with the polariscope, and there is no reason for greater accuracy with one instrument than with the other. In particular, a rather markedly wedge-shaped model can be tolerated since the reflector plates of the interferometer can be adjusted to compensate for such variation in thickness.

[1] F. W. Bubb, "A complete photoelastic instrument," J. Opt. Soc. Am. **30**, 297 (1940).
[2] David Sinclair, "Discussion of 'A complete photoelastic instrument,'" J. Opt. Soc. Am. **30**, 567 (1940).
[3] A. A. Michelson, H. G. Gale, F. Pearson, "Effect of the earth's rotation on the velocity of light," Astrophys. J. **61**, 137–145 (1925).
[4] M. Hetényi, "The fundamentals of three-dimensional photoelasticity," J. App. Mech. **5**, 149 (1938).
[5] R. E. Newton, "A photoelastic study of stresses in rotating disks," J. App. Mech. **7**, 57 (1940).

Unusual Reprinting by Photographs in the Presence of Rare-Earth Mineral Specimens

ASHTON KEITH, SR. AND ASHTON A. KEITH, JR.
Department of Geology, Institute of Sciences, Kansas City, Missouri
June 1, 1941

NUMEROUS photographs of good grade which had been stored face-to-back in a heavy pasteboard box in a cool, dark room, and which had been looked at more than once each year, were found always to remain unchanged and in excellent condition. After looking them over again in June, 1939, and finding no change of any kind, the pictures were replaced in the usual way and the box accidentally was set on top of a collection of mineral specimens from the Barringer Hill pegmatite in the Central Texas Mineral Region where rare-earth minerals were mined by the Westinghouse Company for many years. On happening to examine the photos again in October, 1939, it was found that a number had transferred, or reprinted recognizably clear images on the backs of the cards they faced. No trace of fading or change of any kind could be seen and no unusual attending condition then was found. At that time the possible importance of the discovery was not realized. After putting the photos back as usual, the box again was placed on top of the mineral specimens. Four weeks later, these strange phenomena were mentioned before a small science group and upon opening the box to show samples, indications were found that the

printing was still under way. Off-set images appeared which were less distinct. Two reprints were then placed daily in the full glare of the sun for two weeks without discernible fading. Announcement of this discovery was made before various meetings including the State Academy of Science. Samples also were shown at the joint meeting of the Sections on Physics and Chemistry of the American Association for the Advancement of Science at Columbus, Ohio, December, 1939, where several widely known authorities stated this seems to be something new and worthy of careful investigation. It was generally agreed to be different from the well-known fading of photos under unusual conditions of temperature, etc. On account of our own small laboratory equipment, arrangements accordingly were made for extended investigations at an eastern university and samples of photographs as well as numerous rocks and minerals from the collection were loaned by us for purposes of study, as the writers already had found that when some of the printers were set at a distance from the minerals, there printing soon ceased, while it continued on the others which had been left above the collection of Barringer specimens. Since no definite conclusions have resulted as yet from investigations elsewhere and since interesting data recently have been derived from our simple experiments here, an invitation now is extended to experts who have time and equipment for extended cooperation with us to learn, if possible, the true causes of these phenomena.

Comment on "A Footcandle-Hour Integrator for Daylight" by A. H. Taylor

LLOYD E. VARDEN
Product Information Department, Agfa Ansco, Binghamton, New York
June 9, 1941

MY attention has been called to a statement which appears in the introduction of A. H. Taylor's article, "A footcandle-hour integrator" in the *Journal of the Optical Society of America*, February, 1941. Mr. Taylor says, "*No* photo-tube or other *photoelectric device now available accurately reproduces the spectral sensitivity of the average eye . . .*" (italics mine).

This seems to be a misrepresentation of the facts. Readers of Taylor's article will be surprised at the agreement with the eye sensitivity of the Viscor equipped Photronic cell. This is demonstrated in Fig. 1 (courtesy W. E. I. Company). That this is a fair representation of the cell's characteristics has never been questioned. Beyond all doubt, this Weston cell, which has been commercially available for years, gives an accurate reproduction of the eye sensitivity.

Naturally, if Mr. Taylor means "perfect" when he says "accurate" no photoelectric device does, or ever will, give such a rendering of spectral sensitivity. However, several local members offered unsolicited comments as to the correctness of Taylor's statement, and have since indicated their feeling that the agreement between the two receivers is surely accurate in the accepted photometric sense. Furthermore, it is noteworthy that a Viscor equipped Photronic cell will depart less from the I.C.I. visibility curve than different individual observers.

In conclusion, may I submit the results of a practical test conducted by Electrical Testing Laboratories on a cell with the Viscor filter (Table I).

These measurements were of course made, taking the usual precautions against such sources of error as fatigue, ambient temperature, etc.

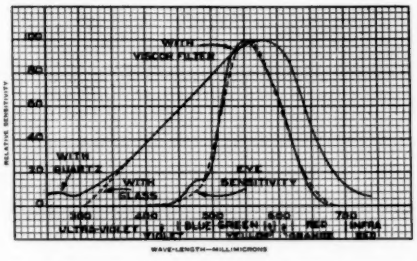

FIG. 1. Spectral sensitivity of Type 1 Photronic cell with various window materials.

TABLE I. *Measurements on Photronic cell response to various illuminants.*

LIGHT SOURCE	PHOTRONIC CELL % CELL RESPONSE	% ERROR
2360	98.9	−1.1
2855	100.0	0
4820	100.0	0
6535	103.2	+2.8
Gas mantle*	92	−8.0
Mercury H.P.*	101	+1.0
Sodium*	97	−3.0
Neon*	100	0
Carbon arc*	94	−6

* The uncertainty in optically determining the luminosity of these sources is much greater than for a tungsten source. The differences between recorded cell outputs are actual differences between cells plus the optical uncertainties in determining luminosity.

TABLE II. *Computed response of Photronic cell to various illuminants having equal luminosity.*

LIGHT SOURCE	CELL WITH VISCOR FILTER	% ERROR
2200° K	99.7	−0.3
2300	99.8	−0.2
2600	100.0	0
2700	100.0	0
2848 (Ill. A)	100.1	+0.1
3000	100.4	+0.4
4000	100.5	+0.5
4800 (Ill. B)	100.7	+0.7
6000	100.7	+0.7
Acetylene	100.7	+0.7
Gas mantle	99.9	−0.1
Sodium	100.4	+0.4
Neon	99.5	−0.5
Mercury		
High pressured	99.3	−0.7
Low pressured	99.4	−0.6
Intr. pressured	99.6	−0.4

Note: Computations were made by multiplying the spectral sensitivity of the average Photronic cell every 10 mμ by the energy for the particular source (based on equal luminosity) and summing up to obtain the total cell response.

However, based on the actual spectral sensitivity of the cell Table II, published by Weston, shows what would appear to be an even higher order of accuracy for a wide range of illuminants. Values of current output are relative to a tungsten source at 2700°K.

Reply to Mr. Varden

A. H. TAYLOR
Physicist, Lamp Department, General Electric Company, Nela Park, Cleveland, Ohio
June 12, 1941

IN reply to Mr. Varden's discussion I would like to make it clear that it was not my intention to disparage the merits of any photoelectric device for the measurement of illumination. My paper was concerned primarily with the use of photo-tubes rather than with photo-e.m.f. cells, since the latter could not be used in my instrument and, so far as I know, no one has developed a suitable filter for any photo-tube to correct its spectral sensitivity even to approximate the adopted visibility curve. Now as regards filter-corrected photo-e.m.f. cells, I know of no published or unpublished data which do not show definite departures from the adopted visibility curve at different parts of the visible spectrum; it is necessary to recognize these departures when working with types of illuminants over a wide range of spectral distribution. The importance of these departures depends upon the use to be made of the data and, therefore, the accuracy required.

I feel, therefore, that the statement to which Mr. Varden takes exception is correct since the sensitivity curves of all filter-corrected cells of which I have knowledge do depart from the adopted visibility curve in certain portions.

Election of Honorary Members of the Optical Society of America

THE Board of Directors of the Optical Society of America at its meeting held in Cambridge, Massachusetts, on February 20, 1941, elected to honorary membership in the Society Professor Theodore Lyman of Harvard University and Sir C. V. Raman of the Indian Institute of Science at Bangalore.

Professor Theodore Lyman

Theodore Lyman received his Doctor's degree from Harvard University in 1900, and soon thereafter began a long membership in the staff of the Physics Department, still happily continuing. For fourteen years he was its Chairman and he is now finishing his thirtieth year as Director of the Jefferson Physical Laboratory. In 1926 his health made it necessary for him to give up some of his teaching duties, but his life-long devotion to spectroscopic research has continued unabated.

This work began with his Ph.D. thesis on what are now known as Lyman ghosts in grating spectra, which he carried through under the direction of Professor W. C. Sabine. His attack on the region of the spectrum discovered by Schumann with a vacuum spectroscope resulted (1906) in accurate wave-length measurements of the lines in the molecular spectrum of hydrogen obtained by an ingenious method of shifted spectra from two slits. Soon after (1914) he found the atomic series of hydrogen which bears his name. A very large number of investigations followed, on the spectra of other elements, the transparency of gases, the reflecting power of solids, etc., etc. During this time he reached shorter and shorter wave-lengths. In 1924 he found the main series of lines in the spectrum of helium, beginning with λ584A. He summarized his own work and that of others in his *Spectroscopy of the Extreme Ultra-Violet*, (1914), of which a revised and extended edition appeared in 1928. In recent years he and his students have attacked by various methods the measurement of the transmission of short waves through air. The latest work reaches to nearly λ100A. Such studies will help in solving the problems of the ionosphere.

His work has always been characterized by the modesty with which it has been presented and the accuracy with which it has been done. Its value has been universally recognized. The respect and affection of his friends has led to his election to the presidency of many societies, scientific and otherwise, of which the American Physical Society and the American Academy of Arts and Sciences might be mentioned. He is also an honorary member of the Royal Institution of London. He was made a Fellow of the National Academy of Sciences in 1917. In 1918 he received the Rumford Medal of the American Academy of Arts and Sciences, in 1930 the Cresson medal of the American Philosophical Society, and in 1931 the Ives medal of the Optical Society.

In more athletic days he made extensive journeys into odd corners of the world in the successful pursuit of big game of various sorts. The bond between him and the Atlantic Salmon (usually a linear one) is still strong and fruitful.

—F. A. Saunders

Professor Theodore Lyman

Sir C. V. Raman

Chandrasekhara Venkata Raman was born at Trichinopoly in Southern India on November 7th, 1888. He received his early education at the Hindu College, Vizagapatam, where his father was a Professor of Mathematics and Physics. Later, he studied for four years at the Presidency College, Madras. He graduated from this institution at the age of sixteen, and two years later took his master's degree, in both cases with the highest distinctions. Even as an undergraduate at the Presidency College, young Raman displayed his genius for original research and independently carried out his first investigations in optics and acoustics which found publication in *Nature* and the *Philosophical Magazine* in 1907. However, in those days there was no scientific career open to any ambitious young men in India so Raman took a competitive examination held by the Government and was appointed as a Gazetted Officer of the Indian Finance Department when barely nineteen years of age.

From June 1907 to July 1917, Raman served as an official of the Government of India, for the major part of which period he was stationed at Calcutta. During these years, he continued his scientific investigations without intermission and published some thirty original communications. His capacity for scientific research thus displayed greatly impressed academic circles at Calcutta and led to his being invited by the University of Calcutta to accept a newly endowed chair of Physics. Besides filling the Palit chair for the next sixteen years, he also held the honorary office of Secretary of the Indian Association for the Cultivation of Science at Calcutta, in the laboratory of which the largest part of his researches was accomplished. In April 1933, Raman left the Calcutta University and accepted a call from the Indian Institute of Science at Bangalore where he now works.

The research papers published in the last 20 years by Raman, and collaborators in his laboratory, include over 600 titles and concern themselves with a variety of subjects such as the dynamics of vibrations and sound, the theory of musical instruments, diffraction problems, meteorological and colloid optics, x-ray diffraction in liquids and solids, magnetism and magneto-crystallic action, electro- and magneto-optics, photoelasticity of solids and fluids, dielectric behavior and ultrasonics. By far the best-known work of Raman is his discovery early in 1928 of the new radiation effect known by his name and which was the natural culmination of the investigations on the scattering of light systematically conducted in his laboratory since 1921. The thirteen years that have elapsed since the discovery of the Raman effect have witnessed the fulfillment of the confidently expressed hope of the discoverer that it would throw new light on many problems in physics and chemistry. Several thousand papers have now appeared on this subject, but the possibilities of further advances in both physical and chemical applications are far from being exhausted.

Sir C. V. Raman

Impressive though his own personal contributions to science have been, Raman's greatest achievement is his leadership in the renaissance of science that has been taking place in India during the past quarter of a century. The progressive enthusiasm for scientific studies and research that exists in India today has largely been inspired by him and encouraged and sus-

tained by his efforts. His personal example of self-dedication to a scientific career, the brilliance and originality of his researches, the international recognition that his work has received, his gift for eloquence which has served to stimulate a wide-spread interest in science, his achievements as a scientific administrator in creating facilities for research and establishing new schools of science and his success in founding journals for publication of scientific work in India are among the factors which have profoundly influenced the progress of science in that country.

The world was not slow to recognize the importance of Raman's achievements as an investigator and a leader of scientific research. Scarcely had he completed six years as a Professor of Physics when the Royal Society of London elected him a Fellow in 1924. He was knighted by the British Government in 1929. He was the recipient of the Matteucci Medal of the Italian Society of Sciences in 1928, of the Hughes Medal of the Royal Society in 1930, and in the same year he was awarded the Nobel Prize for Physics. He has received honorary degrees from a large number of Universities, and is honorary Fellow or Member of many learned societies, both in his native and in foreign countries.

—W. F. MEGGERS

JULY, 1941 J. O. S. A. VOLUME 31

Committees of the Optical Society of America

Committee on Meetings and Programs

A. W. KENNEY, *Chairman*
R. S. ESTEY
L. V. FOSTER
G. E. GRANTHAM
DOROTHY NICKERSON
CLIFTON TUTTLE

Committee on Finance

H. F. KURTZ, *Chairman*
C. L. BAUSCH
L. A. DUBRIDGE
G. R. HARRISON
L. A. JONES
W. F. MEGGERS
J. P. C. SOUTHALL

Frederic Ives Medal Committee

RUDOLF KINGSLAKE, *Chairman*
JESSE W. BEAMS
H. E. IVES
P. E. KLOPSTEG
A. G. WORTHING

Adolph Lomb Medal Committee

W. B. RAYTON, *Chairman*
R. C. GIBBS
G. R. HARRISON
SELIG HECHT
F. L. MOHLER

Membership Committee

DOROTHY NICKERSON, *Chairman*
C. W. GARTLEIN
G. R. HARRISON
A. W. KENNEY
G. S. MONK
BRIAN O'BRIEN
C. F. J. OVERHAGE
F. H. PERRIN
LOUISE SLOAN

Committee on Transfer from Associate to Regular Membership

W. E. FORSYTHE, *Chairman*
H. P. GAGE
R. KINGSLAKE

Committee on Corporation Membership

W. B. RAYTON, *Chairman*
R. C. GIBBS
A. H. BENNETT

Committee on Honorary Membersship

A. G. WORTHING, *Chairman*
J. W. BEAMS
E. C. CRITTENDEN

Committee on Student Membership

BRIAN O'BRIEN, *Chairman*
R. C. GIBBS
G. R. HARRISON
W. B. RAYTON
R. A. SAWYER

Committee on Colorimetry

L. A. JONES, *Chairman*
E. Q. ADAMS
(MRS) BLANCHE BELLAMY
CHARLES BITTINGER
E. C. CRITTENDEN
C. Z. DRAVES
C. E. FOSS
H. P. GAGE
K. S. GIBSON
I. H. GODLOVE
A. C. HARDY
D. B. JUDD
E. M. LOWRY
M. LUCKIESH
D. L. MACADAM
S. M. NEWHALL
DOROTHY NICKERSON
BRIAN O'BRIEN
M. R. PAUL
LOUISE SLOAN
W. B. VAN ARSDEL

Nominating Committee

R. C. GIBBS, *Chairman*
W. E. FORSYTHE
C. E. FOSS
L. V. FOSTER
H. T. WENSEL

Editors of the Journal

G. R. HARRISON, *Editor*

Associate Editors

R. B. BARNES	E. O. HULBURT	LEIGH PAGE
FRANK BENFORD	H. E. IVES	A. H. PFUND
W. R. BRODE	L. A. JONES	W. B. RAYTON
W. G. CADY	D. B. JUDD	R. A. SAWYER
B. H. CARROLL	P. E. KLOPSTEG	E. D. TILLYER
C. J. DAVISSON	W. F. MEGGERS	H. T. WENSEL
K. S. GIBSON	M. G. MELLON	F. E. WRIGHT
A. C. HARDY	G. S. MONK	T. A. WRIGHT

Auditors

C. L. BAUSCH — C. W. FREDERICK

Glossary Committee

A. G. WORTHING, *Chairman*
(Membership of committee not yet appointed.)

Representatives to American Association for Advancement of Science

ELMER HUTCHISSON — SELIG HECHT

Representatives to Division of Physical Sciences of National Research Council

R. C. GIBBS — K. S. GIBSON

Representatives on Inter-Society Color Council

R. C. GIBBS, *Chairman* (voting delegate)

I. H. GODLOVE (voting delegate)	W. C. GRANVILLE
D. B. JUDD (voting delegate)	A. C. HARDY
(MRS.) BLANCHE BELLAMY	L. A. JONES
R. S. ESTEY	DOROTHY NICKERSON
H. P. GAGE	

Representatives on the Governing Board of the American Institute of Physics.

W. B. RAYTON

R. C. GIBBS — A. C. HARDY

Representative on the A. S. A. Sectional Committee on Motion Pictures, Standards for—Z22

B. H. CARROLL

Representative on the A. S. A. Sectional Committee on Safety Glass, Specifications and Methods of Test for—Z26

E. D. TILLYER, Representative
N. F. BARNES, Alternate

Committee on Sponsorship of A. S. A. Committee Z–38, Photography

L. A. JONES, *Chairman* — C. W. GARTLEIN
R. DAVIS — A. C. HARDY

Representative on A. S. A. Sectional Committee Z–38, Photography

BRIAN O'BRIEN

Representative on U. S. N. C. of the International Commission on Illumination

FRANK BENFORD, *Chairman*

L. A. JONES — D. B. JUDD

Representative on American National Committee for International Congresses of Photography

A. C. HARDY

Rochester Section, O. S. A.—Election of Officers

AT the Annual Meeting of the Rochester Section of the Optical Society of America, held on May 27, the following officers were elected to serve until May, 1942:

President:	R. BRUCE HORSFALL
Vice President:	JULIAN H. WEBB
Secretary:	JOHN W. GILLON
Treasurer:	HAROLD W. STRAAT
Councillors:	F. E. ALTMAN
	S. W. BARNES
	L. V. FOSTER
	R. KINGSLAKE
	BRIAN O'BRIEN (*ex officio*)

The report of the Secretary revealed that the membership of the Section now exceeds 200 for the first time, being 204 at present.